A Tableau Algorithm for Description Logics with Nominal Schemas (Technical Report)

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Abstract

We present a tableau-based algorithm to decide concept satisfiability in the description logic $\mathcal{SROIQV}$. The description logic $\mathcal{SROIQV}$ is an extension of the description logic $\mathcal{SROIQ}$, which underlies OWL 2 DL, with the expressive nominal schema construct that enables an integration of rules in such a way that DL-safe Datalog of arbitrary arity are covered within the description logic framework. The tableau-based algorithm provides a basis to implement delayed grounding which was not facilitated by earlier versions of decision procedures for satisfiability in $\mathcal{SROIQV}$.

Introduction

The quest for suitable ontology languages for the Semantic Web (Hitzler, Krötzsch, and Rudolph 2009) has produced a plethora of proposals drawing from all corners of KR research. Most notable among the expressive languages is the W3C[1] standard Web Ontology Language (OWL) (Hitzler et al. 2009), which in its major variant OWL 2 DL is essentially the description logic $\mathcal{SROIQ}(D)$ (Baader et al. 2007). At the same time, logic programming based approaches such as F-Logic (Kifer, Lausen, and Wu 1995) have also been investigated prominently, and have eventually led to the W3C standard Rule Interchange Format (RIF), which in its core variant, called RIF Core (Boley et al. 2010), is essentially Datalog, i.e., function-free Horn logic.

This divergence in underlying paradigms, i.e., description logics on the one hand and logic programming on the other, has naturally led to substantial efforts to reconcile them in a satisfactory manner (see the related work section of (Krisnadhi, Maier, and Hitzler 2011) for a recent survey). However, satisfactory integrations are not easy to obtain, which is mainly due to the fact that description logics are generally supposed to be decidable, and because straightforward integrations with rules lead to undecidability.

The most prominent approach to date to overcome this decidability issue is to alter the semantics of rule bases in such combined languages, and to do this in such a way that variables occurring in rules can bind only to constants which are explicitly present in the knowledge base—in description logic lingua, these variables can bind only to *known individuals*. This reading of (Datalog) rules is only a mild violation of the usual semantics, because such rules are usually interpreted under a Herbrand (minimal model) semantics anyway. Datalog rules with semantics modified such as just mentioned are commonly called *DL-safe Rules*, and decidability is retained if such rules are added to description logic knowledge bases (Motik, Sattler, and Studer 2005).

More recently, however, it was realized that this notion of DL-safety can be relaxed such that only some of the variables in the rules have to be affected (Krötzsch, Rudolph, and Hitzler 2008). And very recently (Krötzsch et al. 2011), the description logic syntax construct *nominal schemas* was introduced, which not only further generalizes the notion of DL-safety, but also enables to express the DL-safe rules (and their generalizations) within the description logic syntax, using the new construct. This new description logic, called $\mathcal{SROIQV}$, in fact has the following features.

- It encompasses $\mathcal{SROIQ}$, which underlies OWL 2 DL.
- It is decidable and has the same worst-case complexity as $\mathcal{SROIQ}$.
- It encompasses DL-safe Datalog.

The latter point was shown in (Krötzsch et al. 2011) for binary Datalog, but this can easily be lifted to predicates of arbitrary arity, as can be seen in the technical report (Carral Martinez et al. 2011).

The decidability proof for $\mathcal{SROIQV}$ from (Krötzsch et al. 2011), however, utilized a rather straightforward and obviously inefficient algorithmization based on *full grounding* (see below). In order to make practical use of the new language, however, it is necessary to develop smarter algorithmizations. So, in this paper, we provide an extension of the $\mathcal{SROIQ}$ tableau algorithm from (Horrocks, Kutz, and Sattler 2006) to handle nominal schemas.

The plan of the paper is as follows. We first recall syntax and semantics of $\mathcal{SROIQV}$. Then we present the tableau algorithm and prove its correctness. Then we conclude.

The Description Logic $\mathcal{SROIQV}$

Syntax. Let N_C, N_R, N_I and N_V be finite and pairwise disjoint sets of *concept names*, *role names*, *individual names*, and *variables*. The set N_R is partitioned into two

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[1] World Wide Web Consortium, http://www.w3.org/

disjoint sets N_R^s of *simple role names* and N_R^n of *non-simple role names*.

Definition 1. *The sets* $\mathbf{R}$ *(*$\mathbf{R}^s$/$\mathbf{R}^n$*) of (*simple/non-simple*)* $\mathcal{SROIQV}$-roles *and* $\mathbf{C}$ *of* $\mathcal{SROIQV}$-concepts *are defined by the following grammar:*

$$\mathbf{R}^s ::= \mathsf{N}_R^s \mid (\mathsf{N}_R^s)^- \mid U \qquad \mathbf{R}^n ::= \mathsf{N}_R^n \mid (\mathsf{N}_R^n)^- \mid U$$

$$\mathbf{R} ::= \mathbf{R}^s \mid \mathbf{R}^n$$

$$\mathbf{C} ::= \top \mid \bot \mid \mathsf{N}_C \mid \{\mathsf{N}_I\} \mid \{\mathsf{N}_V\} \mid \neg\mathbf{C} \mid\mid \mathbf{C} \sqcap \mathbf{C} \mid \mathbf{C} \sqcup \mathbf{C}$$
$$\exists\mathbf{R}.\mathbf{C} \mid \forall\mathbf{R}.\mathbf{C} \mid \exists\mathbf{R}^s.\mathsf{Self} \mid ({\leqslant}n\mathbf{R}^s.\mathbf{C}) \mid ({\geqslant}n\mathbf{R}^s.\mathbf{C})$$

where n is any nonnegative integer. The role U is called the universal role*, while the concept $\top$ and $\bot$ are called* top *and* bottom *concepts. Roles of the form R^- for $R \in \mathsf{N}_R$ are called* inverse roles*. For each $R \in \mathbf{R}$, we define* $\mathsf{Inv}(R) := R^-$ *and* $\mathsf{Inv}(R^-) := R$.

Definition 2. *A* role inclusion axiom (RIA) *is an expression of the form $w \sqsubseteq R$ where $w = R_1 \cdots R_n$ is a string of roles such that either $R \in \mathbf{R}^n$, or $n = 1$ and $R_1 \in \mathbf{R}^s$. A* role disjointness axiom *is an expression of the form* $\mathsf{Dis}(R, S)$ *where $R, S \in \mathbf{R}^s$. An* RBox *is a set of* RBox axioms *which are either an RIA or a role disjointness axiom. An* ABox *is a set of* ABox axioms *which are either of the form $C(a)$, $R(a, b)$, $\neg R(a, b)$, or $a \not\approx b$ where $C \in \mathbf{C}$, $a, b \in \mathsf{N}_I$ and $R \in \mathbf{R}$. A* general concept inclusion *(*GCI*)) is an expression of the form $C \sqsubseteq D$ where $C, D \in \mathbf{C}$. A* TBox *is a set of GCIs. A* knowledge base *is the union $\mathcal{A} \cup \mathcal{R} \cup \mathcal{T}$ of an ABox $\mathcal{A}$, RBox $\mathcal{R}$ and a TBox $\mathcal{T}$.*

Note that the syntax of $\mathcal{SROIQV}$ differs from that of $\mathcal{SROIQ}$ only in the presence of the set N_V of variables and the corresponding class expressions of the form $\{x\}$ for $x \in \mathsf{N}_V$, called *nominal schemas*. They are a kind of variable nominal, i.e., an $x \in \mathsf{N}_V$ should be read as a variable which can bind only to *known* individuals, or in other words, to elements of N_I—and it is in this sense in which nominal schemas generalize the notion of DL-safety discussed in the introduction.

The Grounding Issue. To give an example, consider the $\mathcal{SROIQV}$-axiom in Figure 1. It states that somebody has a conflicting review assignment (paper x) if this person has a paper submitted at the same event z which is co-authored by one of the authors y of paper x. Each of $\{x\}$, $\{y\}$, and $\{z\}$ is a nominal schema, occurring multiple times in the axiom.

The semantics of nominal schemas, which we formally describe below, can alternatively be paraphrased by a transformational approach which eliminates all nominal schemas by *full grounding*: We replace an axiom with nominal schemas by all axioms (without nominal schemas) which can be obtained by substituting all nominal schemas by nominals. The above example (Figure 1) thus gives rise to all axioms of the form given in Figure 2, where a_i, a_j and a_k are individual names. The resulting knowledge base carries the classical DL semantics.

Full grounding results in an increase in the number of axioms which is worst-case exponential in the number of named individuals in the knowledge base. If k is the number of named individuals in the knowledge base, then the axiom in Figure 1 results in k^3 axioms without nominal schemas—the number 3 comes from the fact that the axiom contains 3 distinct nominal schemas. It is thus apparent that an algorithmization of reasoning in $\mathcal{SROIQV}$ based on full grounding will in general not be efficient in practice, and some preliminary experiments reported in (Carral Martinez et al. 2011) confirm this rather obvious insight.

In order to avoid the combinatorial explosion of full grounding, we thus need to establish ways to *delay* grounding and to perform it *selectively*. The contribution of this paper is to show how this can be done in an extension of the $\mathcal{SROIQ}$ tableau algorithm.

Semantics. For completeness and reference, we continue with spelling out the formal semantics of $\mathcal{SROIQV}$. First recall that the original $\mathcal{SROIQ}$ RBox (Horrocks, Kutz, and Sattler 2006) allows for expressing *role characteristics* such as *simple role irreflexivity* $\mathsf{Irr}(S)$, *reflexivity* $\mathsf{Ref}(R)$, *symmetry* $\mathsf{Sym}(R)$, and *transitivity* $\mathsf{Trans}(R)$ where $S \in \mathbf{R}^s$ and $R \in \mathbf{R}$. From the semantics defined below, each of those can respectively be expressed with the following (sets of) TBox and/or RBox axioms: $\top \sqsubseteq \neg\exists S.\mathsf{Self}$; $\{\top \sqsubseteq \exists S_{\mathsf{aux}}.\mathsf{Self}, S_{\mathsf{aux}} \sqsubseteq R\}$ with some fresh simple role S_{aux}; $\mathsf{Inv}(R) \sqsubseteq R$; and $RR \sqsubseteq R$. In addition, *asymmetric role* can be expressed by the disjointness axiom $\mathsf{Dis}(R, \mathsf{Inv}(R))$.

Definition 3. *An set of RIAs is* regular *if there is a strict partial order $\prec$ on $\mathbf{R}$ such that for every role $R, S, S_i \in \mathbf{R}$ (i) if $R \notin \{S, \mathsf{Inv}(S)\}$, then $S \prec R$ iff $\mathsf{Inv}(S) \prec R$; and (ii) every RIA is of the form $RR \sqsubseteq R$, $\mathsf{Inv}(R) \sqsubseteq R$, $RS_1 \cdots S_k \sqsubseteq R$, $S_1 \cdots S_k R \sqsubseteq R$, or $S_1 \cdots S_k \sqsubseteq R$ where $S_i \prec R$ for $1 \leqslant i \leqslant k$. An RBox is* regular *iff the set containing of all of its RIAs is regular. A knowledge base is regular iff its RBox component is regular.*

From now on, we always assumed that knowledge bases are regular.

Given a regular RBox $\mathcal{R}$, let $\mathcal{R}_h$ the largest subset of $\mathcal{R}$ containing only RIAs of the form $R \sqsubseteq S$ where R is a string of role of length 1. The relation $\sqsubseteq^{*}$ is defined to be the transitive-reflexive closure of $\sqsubseteq$ over the set $\mathcal{R}_h \cup \{\mathsf{Inv}(R) \sqsubseteq \mathsf{Inv}(S) \mid R \sqsubseteq S \in \mathcal{R}_h\}$.

We denote the set of variables occurring (as nominal schemas, possibly multiple times) in a concept C as $\mathsf{Var}(C)$. Likewise, the set of individual names occurring (as nominals) in a concept C is denoted by $\mathsf{Ind}(C)$. Both sets are naturally extended to knowledge base axioms, and sets of concepts. Given a knowledge base and a concept, we also assume that N_V (resp. N_I) contains precisely all variables (resp. individual names) occurring either in the knowledge base or in the concept in the context. This makes every v-assignment, as defined below, map variables to explicitly known individual names.

Definition 4. *A* (full) v-assignment *is a total function from N_V to N_I. Let $M \subseteq \mathsf{N}_V$ be a set of variables and $\mathcal{Z}$ be a v-assignment. Then $\mathcal{Z}_M$, called M-*restricted v-assignment*, is obtained from $\mathcal{Z}$ by restricting its domain to M. In particular, $\mathcal{Z}_\emptyset$ is called the* empty v-assignment*.*

Let m be the cardinality of N_V and n the cardinality of N_I. Then there are n^m different full v-assignments relevant

$$\begin{aligned}&\exists \text{hasReviewAssignment}.((\{x\} \sqcap \exists \text{hasAuthor}.\{y\}) \sqcap (\{x\} \sqcap \exists \text{atVenue}.\{z\})) \sqcap \\ &\qquad \sqcap \exists \text{hasSubmittedPaper}.(\exists \text{hasAuthor}.\{y\} \sqcap \exists \text{atVenue}.\{z\}) \sqsubseteq \exists \text{hasConflictingAssignedPaper}.\{x\}\end{aligned}$$

Figure 1: Example $\mathcal{SROIQV}$-axiom from (Krisnadhi, Maier, and Hitzler 2011).

$$\begin{aligned}&\exists \text{hasReviewAssignment}.((\{a_i\} \sqcap \exists \text{hasAuthor}.\{a_j\}) \sqcap (\{a_i\} \sqcap \exists \text{atVenue}.\{a_k\})) \\ &\qquad \sqcap \exists \text{hasSubmittedPaper}.(\exists \text{hasAuthor}.\{a_j\} \sqcap \exists \text{atVenue}.\{a_k\}) \sqsubseteq \exists \text{hasConflictingAssignedPaper}.\{a_i\}\end{aligned}$$

Figure 2: For each choice of $a_i, a_j, a_k \in \mathsf{N}_I$, the axiom above is a grounding of the axiom from Figure 1.

to our discussion. From now on, every v-assignment is considered as full v-assignment, unless stated otherwise.

Definition 5. *Let C be a $\mathcal{SROIQV}$ concept, and $\mathcal{Z}$ a (possibly M-restricted) v-assignment. Then, $\mathsf{gr}_{\mathcal{Z}}(C)$ is the concept obtained from C by replacing occurrences of every variable $x \in \mathsf{Var}(C)$ with $\mathcal{Z}(x)$. Moreover, for a set of concepts Σ, $\mathsf{gr}_{\mathcal{Z}}(\Sigma) = \{\mathsf{gr}_{\mathcal{Z}}(C) \mid C \in \Sigma\}$.*

Let $M \subseteq \mathsf{N}_V$ be a set of variables. Definition 5 implies that $\mathsf{gr}_{\mathcal{Z}_M}(C)$ retains the occurrences of some variables (as nominal schemas) whenever $\mathsf{Var}(C) \setminus M \neq \emptyset$. In particular, if $\mathsf{Var}(C) \cap M = \emptyset$, then $\mathsf{gr}_{\mathcal{Z}_M}(C) = C$. When both $\mathsf{Var}(C) \setminus M \neq \emptyset$ and $\mathsf{Var}(C) \cap M \neq \emptyset$, we say that $\mathsf{gr}_{\mathcal{Z}_M}(C)$ is *partially grounded*. On the other hand, if $\mathsf{Var}(C) \subseteq M$, then $\mathsf{gr}_{\mathcal{Z}_M}(C)$ contains no occurrence of variables, and we say that it is *(completely) grounded*. Obviously, $\mathsf{gr}_{\mathcal{Z}}(C)$ is completely grounded for any full v-assignment $\mathcal{Z}$.

Definition 6. *An* interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *consists of a nonempty set $\Delta^{\mathcal{I}}$ called the* domain, *and a function $\cdot^{\mathcal{I}}$ which maps each element $a \in \mathsf{N}_I$, $A \in \mathsf{N}_C$, and $R \in \mathsf{N}_R$ resp. to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Given a v-assignment $\mathcal{Z}$ and an interpretation $\mathcal{I}$ as defined above, the semantics of $\mathcal{SROIQV}$ concepts, roles, individual names is defined w.r.t. $\mathcal{Z}$ according to the function $\cdot^{\mathcal{I},\mathcal{Z}}$ as in Figure 3, where $a \in \mathsf{N}_I$, $A \in \mathsf{N}_C$, $R \in \mathsf{N}_R$, $v \in \mathsf{N}_V$, $t \in \mathsf{N}_I \cup \mathsf{N}_V$, $C, D \in \mathbf{C}$: Given an interpretation $\mathcal{I}$ and a v-assignment $\mathcal{Z}$, $\mathcal{I}$ and $\mathcal{Z}$* satisfy *an axiom α in a knowledge base, written $\mathcal{I}, \mathcal{Z} \models \alpha$, iff the following conditions are satisfied:*

$$\mathcal{I}, \mathcal{Z} \models C(a) \textit{ iff } a^{\mathcal{I},\mathcal{Z}} \in C^{\mathcal{I},\mathcal{Z}}$$

$$\mathcal{I}, \mathcal{Z} \models R(a,b) \textit{ iff } \langle a^{\mathcal{I},\mathcal{Z}}, b^{\mathcal{I},\mathcal{Z}} \rangle \in R^{\mathcal{I},\mathcal{Z}}$$

$$\mathcal{I}, \mathcal{Z} \models \neg R(a,b) \textit{ iff } \langle a^{\mathcal{I},\mathcal{Z}}, b^{\mathcal{I},\mathcal{Z}} \rangle \notin R^{\mathcal{I},\mathcal{Z}}$$

$$\mathcal{I}, \mathcal{Z} \models a \neq b \textit{ iff } a^{\mathcal{I},\mathcal{Z}} \neq b^{\mathcal{I},\mathcal{Z}}$$

$$\mathcal{I}, \mathcal{Z} \models C \sqsubseteq D \textit{ iff } C^{\mathcal{I},\mathcal{Z}} \subseteq D^{\mathcal{I},\mathcal{Z}}$$

$$\mathcal{I}, \mathcal{Z} \models \textsf{Dis}(R,S) \textit{ iff } R^{\mathcal{I},\mathcal{Z}} \cap S^{\mathcal{I},\mathcal{Z}} = \emptyset$$

$$\mathcal{I}, \mathcal{Z} \models R_1 \cdots R_n \sqsubseteq R \textit{ iff } (R_1 \cdots R_n)^{\mathcal{I},\mathcal{Z}} \subseteq R^{\mathcal{I},\mathcal{Z}}$$

where $(R_1 \cdots R_n)^{\mathcal{I},\mathcal{Z}} = R_1^{\mathcal{I},\mathcal{Z}} \circ \cdots \circ R_n^{\mathcal{I},\mathcal{Z}}$ with '$\circ$' denotes the usual binary composition of relations.

Let $KB = \mathcal{A} \cup \mathcal{R} \cup \mathcal{T}$ be a knowledge base as defined in Definition 2. We further say that $\mathcal{I}$ and $\mathcal{Z}$ satisfy *KB, written $\mathcal{I}, \mathcal{Z} \models KB$, if $\mathcal{I}, \mathcal{Z} \models \alpha$ for every $\alpha \in KB$. We also say that $\mathcal{I}$ and $\mathcal{Z}$* satisfy *a concept C w.r.t. KB if $C^{\mathcal{I},\mathcal{Z}} \neq \emptyset$ and $\mathcal{I}, \mathcal{Z} \models KB$. Next, $\mathcal{I}$* satisfies *C w.r.t. KB if $\mathcal{I}$ and $\mathcal{Z}$ satisfy C w.r.t. KB for every v-assignment $\mathcal{Z}$. If such an interpretation $\mathcal{I}$ exists, we say that C is* satisfiable *w.r.t. KB.*

$$\begin{aligned}
a^{\mathcal{I},\mathcal{Z}} &= a^{\mathcal{I}} \quad A^{\mathcal{I},\mathcal{Z}} = A^{\mathcal{I}}, \quad R^{\mathcal{I},\mathcal{Z}} = R^{\mathcal{I}} \\
v^{\mathcal{I},\mathcal{Z}} &= (\mathcal{Z}(v))^{\mathcal{I},\mathcal{Z}} = (\mathcal{Z}(v))^{\mathcal{I}}, \quad (\{t\})^{\mathcal{I},\mathcal{Z}} = \{t^{\mathcal{I},\mathcal{Z}}\} \\
\top^{\mathcal{I},\mathcal{Z}} &= \top^{\mathcal{I}} = \Delta^{\mathcal{I}}, \quad \bot^{\mathcal{I},\mathcal{Z}} = \bot^{\mathcal{I}} = \emptyset \\
(\neg C)^{\mathcal{I},\mathcal{Z}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I},\mathcal{Z}} \\
(C \sqcap D)^{\mathcal{I},\mathcal{Z}} &= C^{\mathcal{I},\mathcal{Z}} \cap D^{\mathcal{I},\mathcal{Z}}, \ (C \sqcup D)^{\mathcal{I},\mathcal{Z}} = C^{\mathcal{I},\mathcal{Z}} \cup D^{\mathcal{I},\mathcal{Z}} \\
(\exists R.\mathsf{Self})^{\mathcal{I},\mathcal{Z}} &= \{\delta \mid \langle \delta, \delta \rangle \in R^{\mathcal{I},\mathcal{Z}}\} \\
(\exists R.C)^{\mathcal{I},\mathcal{Z}} &= \{\delta \mid \exists \epsilon : \langle \delta, \epsilon \rangle \in R^{\mathcal{I}} \text{ and } \delta' \in C^{\mathcal{I},\mathcal{Z}}\} \\
(\forall R.C)^{\mathcal{I},\mathcal{Z}} &= \{\delta \mid \forall \epsilon : \text{ if } \langle \delta, \epsilon \rangle \in R^{\mathcal{I}} \text{ then } \epsilon \in C^{\mathcal{I},\mathcal{Z}}\} \\
(\geqslant n R.C) &= \{\delta \mid \#\{\langle \delta, \epsilon \rangle \in R^{\mathcal{I},\mathcal{Z}} \mid \epsilon \in C^{\mathcal{I},\mathcal{Z}}\} \geqslant n\} \\
(\leqslant n R.C) &= \{\delta \mid \#\{\langle \delta, \epsilon \rangle \in R^{\mathcal{I},\mathcal{Z}} \mid \epsilon \in C^{\mathcal{I},\mathcal{Z}}\} \leqslant n\} \\
U^{\mathcal{I},\mathcal{Z}} &= U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \\
(R^-)^{\mathcal{I},\mathcal{Z}} &= (R^-)^{\mathcal{I}} = \{\langle \delta, \delta' \rangle \mid \langle \delta', \delta \rangle \in R^{\mathcal{I}}\}
\end{aligned}$$

Figure 3: Interpretations, as in Definition 6.

The following lemma is straightforward.

Lemma 7. *Let C be a $\mathcal{SROIQV}$ concept, $\mathcal{I}$ be an interpretation, $\Gamma \subseteq \mathsf{N}_V$ be a set of variables, and $s \in \Delta^{\mathcal{I}}$. Then, for every v-assignment $\mathcal{Z}$, $s \in (\mathsf{gr}_{\mathcal{Z}}(C))^{\mathcal{I},\mathcal{Z}}$ iff $s \in (\mathsf{gr}_{\mathcal{Z}_\Gamma}(C))^{\mathcal{I},\mathcal{Z}}$ iff $s \in C^{\mathcal{I},\mathcal{Z}}$.*

A Tableau Algorithm for $\mathcal{SROIQV}$

Our algorithm is based on the $\mathcal{SROIQ}$ tableau algorithm from (Horrocks, Kutz, and Sattler 2006), modified where appropriate.

Reduction of Inference Problems. Satisfiability of concepts w.r.t. an ABox, TBox and regular RBox can be reduced to satisfiability of concepts w.r.t. a regular RBox in which no universal role occurs. This is a well known result for $\mathcal{SROIQ}$ (Horrocks, Kutz, and Sattler 2006). This result can be easily extended for $\mathcal{SROIQV}$.

Let C be a concept, $\mathcal{A}$ an ABox, $\mathcal{T}$ a TBox and $\mathcal{R}$ a regular RBox. Note that the universal role is disallowed in $\mathcal{R}$. W.l.o.g., we assume that no nominal schema occurs in two different axioms in $\mathcal{A} \cup \mathcal{T}$, i.e., if a nominal schema $\{x\}$ occurs (possibly multiple times) in an axiom $\alpha \in \mathcal{A} \cup \mathcal{T}$, then $\{x\}$ does not occur in other axioms $\beta \in \mathcal{A} \cup \mathcal{T}$ with $\beta \neq \alpha$. This can be obtained easily by appropriate renaming of nominal schemas if the assumption is violated. To internalize $\mathcal{A}$, we first associate for each individual name $c \in \mathsf{Ind}(\mathcal{A})$, a new nominal o_c not appearing in $\mathcal{T}$ and C. Then, C is satisfiable w.r.t. $\mathcal{A} \cup \mathcal{R} \cup \mathcal{T}$ iff $C \sqcap \bigsqcap_{\alpha \in \mathcal{A}} \exists U.C_\alpha$ is satisfiable w.r.t. $\mathcal{R} \cup \mathcal{T}$ where $C_{D(a)} := o_a \sqcap D$; $C_{R(a,b)} := o_a \sqcap \exists R.o_b$; $C_{\neg R(a,b)} := o_a \sqcap \forall R.\neg o_b$; and $C_{a \dot{\neq} b} := o_a \sqcap \neg o_b$.

Next, assume that $\mathcal{A}$ has been internalized. Let C be a concept , $\mathcal{T}$ a TBox and $\mathcal{R}$ a regular RBox. The TBox $\mathcal{T}$ and the universal role U can be internalized as follows. Let $U' \neq U$ be a new role not occurring in C, $\mathcal{T}$ and $\mathcal{R}$. We have, as an obvious corollary of Lemma 8 from (Horrocks, Kutz, and Sattler 2006), that C is satisfiable w.r.t. $\mathcal{R} \cup \mathcal{T}$ iff $C' \sqcap C_{\mathcal{T}'}$ is satisfiable w.r.t. $\mathcal{R}'$ where S_{aux} is a fresh simple role name not occurring in $\mathcal{T} \cup \mathcal{R}$, and:

(i) $\mathcal{R}' := \mathcal{R} \cup \{U'U' \sqsubseteq U', \mathsf{Inv}(U') \sqsubseteq U', S_{\mathsf{aux}} \sqsubseteq U'\}$
$\cup \{R \sqsubseteq U' \mid R \neq U, R \text{ occurs in } C, \mathcal{T}, \text{ or } \mathcal{R}\}$;
(ii) C' is obtained from C by replacing U with U';
(iii) $\mathcal{T}'$ is obtained from $\mathcal{T} \cup \{\top \sqsubseteq \exists S_{\mathsf{aux}}.\mathsf{Self}\}$ by replacing U with U';
(iv) $C_{\mathcal{T}'} := \exists U'.\{z\} \sqcap \forall U'.\bigsqcap_{D \sqsubseteq E \in \mathcal{T}'}(\neg D \sqcup E)$ with $\{z\}$ a new nominal schema not appearing in C' and $\mathcal{T}'$.

In the above transformation, the RBox $\mathcal{R}'$ is said to be *reduced*. From now on, every (regular) RBox in our discussion is assumed to be reduced.

The Algorithm. For a regular RBox $\mathcal{R}$ and a (possibly inverse) role S occurring in $\mathcal{R}$, we define a non-deterministic finite automaton (NFA) $\mathcal{B}_S$ that contains all consequences of $\mathcal{R}$ resulting from inclusions between role chains (i.e., path of roles) and S. Note that nominal schemas do not occur in the RBox, hence the NFA construction is exactly the same as for $\mathcal{SROIQ}$. We will recall the key results here and refer for details to (Horrocks and Sattler 2004). As usual, ε will denote the empty word.

Theorem 8. *Let $\mathcal{R}$ be a regular (reduced) RBox and $\mathcal{R}_h = \{w \sqsubseteq R \in \mathcal{R} \mid w = R_1 \cdots R_n, n \geqslant 0\}$. Then $\mathcal{I}$ is a model of $\mathcal{R}_h$ if and only if, for every (possibly inverse) role S occurring in $\mathcal{R}_h$, every word $w \in L(\mathcal{B}_S)$ (i.e., in the language which $\mathcal{B}_S$ accepts), and every pair $\langle x, y\rangle \in w^{\mathcal{I}}$, it holds that $\langle x, y\rangle \in S^{\mathcal{I}}$.*

The technical lemma below is also from (Horrocks and Sattler 2004).

Lemma 9. *Let $\mathcal{R}$ be a regular RBox. Then:*
1. *$S \in L(\mathcal{B}_S)$, and, if $w \sqsubseteq S \in \mathcal{R}$, then $w \in L(\mathcal{B}_S)$;*
2. *$L(\mathcal{B}_S) \supseteq \{R \mid R \mathrel{\underline{\boxast}} S\}$, and if S is a simple role, then $L(\mathcal{B}_S) = \{R \mid R \mathrel{\underline{\boxast}} S\}$;*
3. *$L(\mathcal{B}_{\mathsf{Inv}(S)}) = \{\mathsf{Inv}(w) \mid w \in L(\mathcal{B}_S)\}$;*

We assume that all concepts are in negation normal form, i.e., negation only occurs in front of concept names or in front of $\exists R.\mathsf{Self}$. We denote $\mathsf{NNF}(C)$ for the concept obtained by transforming C into negation normal form. This transformation can be done in linear time. We also define the set $\mathsf{clos}(C)$ as the smallest set containing C that is closed under sub-concepts and NNF. The set $\mathsf{fcl}(C, \mathcal{R})$ is thus defined for a $\mathcal{SROIQV}$ concept C and RBox $\mathcal{R}$ as

$$\begin{aligned}\mathsf{fcl}(C, \mathcal{R}) &:= \mathsf{clos}(C) \\ &\quad \cup \{\forall \mathcal{B}_R(q).D \mid \forall R.D \in \mathsf{clos}(C), q \in Q(\mathcal{B}_R)\}\end{aligned}$$

where for an NFA $\mathcal{B}$ and a state q of $\mathcal{B}$, $Q(\mathcal{B}_R)$ is the set of states of $\mathcal{B}_R$, and $\mathcal{B}(q)$ is the NFA obtained from $\mathcal{B}$ by fixing q as the only initial state of $\mathcal{B}$. In addition, we define $\mathsf{rol}(C, \mathcal{R})$ as the set of role names and their inverses occurring in C and $\mathcal{R}$, and the notation $p \xrightarrow{R} q \in \mathcal{B}$ means that there is a transition from a state p to a state q in $\mathcal{B}$ with the symbol R. Let Φ be the set of all full v-assignments from N_V to N_I. We also define the following sets for a given concept C and RBox $\mathcal{R}$:

$$\begin{aligned}\mathsf{qcl}_0(C, \mathcal{R}) &:= \{(\leqslant kR.C) \mid (\leqslant nR.C) \in \mathsf{fcl}(C, \mathcal{R}), k \leqslant n\} \\ \mathsf{qcl}(C, \mathcal{R}) &:= \mathsf{qcl}_0(C, \mathcal{R}) \cup \{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \mid C \in \mathsf{qcl}_0(C, \mathcal{R}), \\ &\qquad\qquad \Gamma \subseteq \mathsf{N}_V, \mathcal{Z} \in \Phi\} \\ \mathsf{gcl}(C, \mathcal{R}) &:= \{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \mid C \in \mathsf{fcl}(C, \mathcal{R}), \Gamma \subseteq \mathsf{N}_V, \mathcal{Z} \in \Phi, \\ &\qquad\qquad \mathsf{gr}_{\mathcal{Z}_\Gamma}(C, \mathcal{R}) \notin \mathsf{fcl}(C, \mathcal{R})\}\end{aligned}$$

Let C_0 be a $\mathcal{SROIQV}$ concept and $\mathcal{R}$ a reduced RBox. Also, let $\mathbf{N}_o = \{o_1, \ldots, o_\ell\}$ be the set of all nominals occurring in C_0 (thus of the same cardinality as N_I), and $\mathbf{N}_s = \{v_1, \ldots, v_m\}$ be the set of all nominal schemas occurring in C_0. In addition, let $\mathbf{N}_a$ be a possibly infinite set of *auxiliary nominals* that is disjoint with both $\mathbf{N}_o$ and $\mathbf{N}_s$. W.l.o.g., we assume that $\mathbf{N}_o \neq \emptyset$, i.e., there is at least one individual name occurring in C_0.

A *completion graph* for C_0 w.r.t. $\mathcal{R}$ is a directed graph $\mathbf{G} = (V, E, \mathcal{L}, \dot{\neq})$ such that

- each node $x \in V$ is labeled with a set
$$\mathcal{L}(x) \subseteq \mathsf{fcl}(C_0, \mathcal{R}) \cup \mathbf{N}_a \cup \mathsf{qcl}(C_0, \mathcal{R}) \cup \mathsf{gcl}(C_0, \mathcal{R})$$
- each edge $\langle x, y\rangle \in E$ is labeled with a set $\mathcal{L}(\langle x, y\rangle)$ of (possibly inverse) roles occurring in C_0 and $\mathcal{R}$;
- $\dot{\neq}$ is a symmetric binary inequality relation on V.

Also, note that $\mathbf{N}_s \cup \mathbf{N}_o \subseteq \mathsf{fcl}(C_0, \mathcal{R})$ already. In what follows, the notation $R \in \mathcal{L}(\langle x, y\rangle)$ stands for $\langle x, y\rangle \in E$ and $R \in \mathcal{L}(\langle x, y\rangle)$.

Let $S \in \mathcal{L}(\langle x, y\rangle)$ for some nodes x and y. Then for every role R such that $S \mathrel{\underline{\boxast}} R$, we say that y is an *R-successor* of x and x is an *R-predecessor* of y. When the role name does not matter, we simply say that y is a successor of x and x is a predecessor of y. A node y is a *(R-)neighbor* of x if y is an (R-)successor of x or x is an ($\mathsf{Inv}(R)$-)successor of y. The *descendant* relation is the transitive closure of predecessor, whereas *ancestor* is the transitive closure of successor.

A node $x \in V$ of $\mathbf{G}$ is a *nominal node* if $o \in \mathcal{L}(x)$ for some $o \in \mathbf{N}_o \cup \mathbf{N}_a$. In particular, x is an *original* nominal node if $\mathcal{L}(x)$ contains a nominal $o \in \mathbf{N}_o$. We say that an element $o \in \mathbf{N}_o \cup \mathbf{N}_a$ is *new in* $\mathbf{G}$ if $o \notin \mathcal{L}(x)$ for every node x of $\mathbf{G}$. A *blockable node* is a node in $\mathbf{G}$ that is not a nominal node. Also, an R-neighbor y of a node x is *safe* if either x is blockable or x is a nominal node and y is not blocked.

A node x is *directly blocked* if x has ancestors x', y, and y' such that

1. x is a successor of x' and y is a successor of y';
2. x and y are blockable and the path from y to x consists only of blockable nodes;
3. $\mathcal{L}(x) = \mathcal{L}(y)$, and $\mathcal{L}(x') = \mathcal{L}(y')$; and
4. $\mathcal{L}(\langle x', x\rangle) = \mathcal{L}(\langle y', y\rangle)$.

Here, we say that the node y *blocks* x. A node is *blocked* if either it is directly blocked or it is blockable and its predecessor is blocked. A blockable node x is *indirectly blocked* if its predecessor is blocked.

Given a role R, a node x and a concept C, we define

$$R^{\mathbf{G}}(x, C) = \{y \mid y \text{ is an } R\text{-neighbor of } x,\ C \in \mathcal{L}(y)\}$$

We say that $\mathbf{G}$ *contains a clash* whenever there are nodes x and y such that at least one of the following holds:

(1) $\bot \in \mathcal{L}(x)$;
(2) $\{A, \neg A\} \subseteq \mathcal{L}(x)$ for some $A \in \mathsf{N}_C$;
(3) x is an R-neighbor of x and $\neg R.\mathsf{Self} \in \mathcal{L}(x)$;
(4) $\mathsf{Dis}(R, S) \in \mathcal{R}$ for some role R and S, and y is both an R- and S- neighbor of x;
(5) there is some concept $(\leqslant nR.C) \in \mathcal{L}(x)$, and $\{z_0, \ldots, z_n\} \subseteq R^{\mathbf{G}}(x, C)$ with $z_i \neq z_j$ for all $0 \leqslant i < j \leqslant n$;
(6) $o \in \mathcal{L}(x) \cap \mathcal{L}(y)$ for some $o \in \mathbf{N}_o \cup \mathbf{N}_a$, but $x \neq y$.

We initially define an *initial* completion graph $\mathbf{G} = (V, \emptyset, \mathcal{L}, \emptyset)$ where $V = \{r_0, r_1, \ldots, r_\ell\}$, $\mathcal{L}(r_0) = \{C_0, \top\}$, $\mathcal{L}(r_i) = \{o_i, \top\}$ for $1 \leqslant i \leqslant \ell$.

The operation COPYEDGE$(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle)$ on a completion graph $\mathbf{G} = (V, E, \mathcal{L}, \neq)$ copies labels of an edge $\langle x_1, y_1\rangle$ to another edge $\langle x_2, y_2\rangle$, taking into account whether $\langle x_2, y_2\rangle$ or its inverse already exists in E. It is defined as follows:

1. if $\langle x_2, y_2\rangle \in E$, then $\mathcal{L}(\langle x_2, y_2\rangle) := \mathcal{L}(\langle x_2, y_2\rangle) \cup \mathcal{L}(\langle x_1, y_1\rangle)$;
2. else if $\langle y_2, x_2\rangle \in E$, then $\mathcal{L}(\langle y_2, x_2\rangle) := \mathcal{L}(\langle y_2, x_2\rangle) \cup \{\mathsf{Inv}(R) \mid R \in \langle x_1, y_1\rangle\}$;
3. else $E := E \cup \{\langle x_2, y_2\rangle\}$ and $\mathcal{L}(\langle x_2, y_2\rangle) := \mathcal{L}(\langle x_1, y_1\rangle)$.

The operation PRUNE(x) on a node x in a completion graph $\mathbf{G} = (V, E, \mathcal{L}, \neq)$ obtains a new graph from $\mathbf{G}$ by removing x and, recursively, all of its blockable successors. It is defined as follows:

1. for every successor y of x, remove $\langle x, y\rangle$ from E; and, if y is blockable, then PRUNE(y);
2. remove x from V.

The operation MERGE(x, y) merges a node x into a node y in $\mathbf{G} = (V, E, \mathcal{L}, \neq)$ yielding a new graph that is obtained from $\mathbf{G}$ by the following steps.

1. For each edge $\langle z, x\rangle$, first, do COPYEDGE$(\langle z, x\rangle, \langle z, y\rangle)$, and then remove $\langle z, x\rangle$ from E.
2. For each edge $\langle x, z\rangle$ where z is either a nominal node or a nominal schema node, do COPYEDGE$(\langle x, z\rangle, \langle y, z\rangle)$, and then remove $\langle x, z\rangle$ from E.
3. Set $\mathcal{L}(y) := \mathcal{L}(y) \cup \mathcal{L}(x)$.
4. Add $y \neq z$ for every z with $x \neq z$.
5. PRUNE(x).

We now have almost all preparations needed for specifying the algorithm. The following Definition is key to our realization of delayed grounding.

Definition 10. *The set* $\mathsf{tv}(C)$ *for a concept* C*, called the* top level *of* C*, is defined inductively as: (i)* $\mathsf{tv}(A) = \mathsf{tv}(\neg A) = \{A\}$ *for* A *atomic or* $\exists R.\mathsf{Self}$*; (ii)* $\mathsf{tv}(C) = \mathsf{tv}(C_1) \cup \mathsf{tv}(C_2)$ *for* $C = C_1 \sqcap C_2$ *or* $C = C_1 \sqcup C_2$*; and (iii)* $\mathsf{tv}(C) = \emptyset$*, otherwise.*

The notion of top level just defined serves as a guard for some tableau expansion rules, such that it essentially forces a grounding in some cases before expansion. This is required for correctness of the algorithm.

The expansion rules, as given in Figures 4 and 5, are now applied exhaustively, starting from the initial completion graph for C_0 and $\mathcal{R}$, while obeying the following prioritization—i.e., rules with higher priority must be applied first. In order to specify the priorities, we require the notion of *level* of a nominal node x, defined inductively as follows.

- If $o \in \mathcal{L}(x)$ for some $o \in \mathbf{N}_o$, then x is of level 0.
- If $\mathcal{L}(x) \cap \mathbf{N}_o = \emptyset$, then the level of x is $1 + k$, where k is the smallest of all levels of neighbors of x.

Note that, initially, all the nominal nodes contained in $\mathbf{G}$ are of level 0. The tableau expansion rules are now prioritized as follows.

1. The o-rule is applied with the highest priority.
2. The $\leqslant$- and NN-rules are applied with the second highest priority; their application is done first to nodes with lower levels before higher level nodes; the NN-rule takes precedence over the $\leqslant$-rule when they are both applicable to the same node.
3. All other rules are applied with a lower priority.

A completion graph is *complete* if it contains a clash, or when none of the rules is applicable. The algorithm returns "C_0 is satisfiable w.r.t. $\mathcal{R}$" if the expansion rules can be applied to C_0 and $\mathcal{R}$ in such a way that it results in a complete, clash-free completion graph. Otherwise, it returns "C_0 is unsatisfiable w.r.t. $\mathcal{R}$."

Difference to $\mathcal{SROIQ}$. Let us dwell for a moment on some of the differences between our algorithm and the $\mathcal{SROIQ}$-algorithm from (Horrocks, Kutz, and Sattler 2006) on which it is based. The most obvious difference is the new rules G1, G2, and G3, which realize delayed and selective grounding, as we will demonstrate with an example in the next section.

Another, more subtle difference, is the introduction of a precondition to some expansion rules, which utilizes the concept top level notion from Definition 10. Essentially, it forces that in some cases nominal schemas are grounded before they are propagated throughout the graph, and this features prominently in some parts of the correctness proofs. As such, it is a technical notion which seems required for algorithm correctness, but at this stage we are unable to provide much in terms of deeper insights or intuitions, other than to refer to the proofs.

$\sqcap$-rule: if	$C_1 \sqcap C_2 \in \mathcal{L}(x)$, x is not indirectly blocked, $\mathsf{tv}(C_1 \sqcap C_2) \cap \mathbf{N}_s = \emptyset$, and $\{C_1, C_2\} \not\subseteq \mathcal{L}(x)$,
then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{C_1, C_2\}$
$\sqcup$-rule: if	$C_1 \sqcup C_2 \in \mathcal{L}(x)$, x is not indirectly blocked, $\mathsf{tv}(C_1 \sqcup C_2) \cap \mathbf{N}_s = \emptyset$, and $\{C_1, C_2\} \cap \mathcal{L}(x) = \emptyset$,
then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{D\}$ for some $D \in \{C_1, C_2\}$
sr-rule: if	$\exists R.\mathsf{Self} \in \mathcal{L}(x)$, x is not blocked, $R \notin \mathcal{L}(\langle x, x\rangle)$,
then	create a new edge $\langle x, x\rangle$ (whose set of label is empty), if not exists, and set $\mathcal{L}(\langle x, x\rangle) := \mathcal{L}(\langle x, x\rangle) \cup \{R\}$
$\exists$-rule: if	$\exists R.C \in \mathcal{L}(x)$, x is not blocked, $\mathsf{tv}(C) \cap \mathbf{N}_s = \emptyset$, and x has no R-neighbor y such that $C \in \mathcal{L}(y)$,
then	create a new node y with $\mathcal{L}(\langle x, y\rangle) := \{R\}$ and $\mathcal{L}(y) := \{C, \top\}$
$\forall_1$-rule: if	$\forall R.C \in \mathcal{L}(x)$, x is not indirectly blocked, and $\forall \mathcal{B}_R.C \notin \mathcal{L}(x)$,
then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{\forall \mathcal{B}_R.C\}$
$\forall_2$-rule: if	$\forall \mathcal{B}(p).C \in \mathcal{L}(x)$, $p \xrightarrow{R} q \in \mathcal{B}(p)$, x is not indirectly blocked, x has an R-neighbor y such that $\forall \mathcal{B}(q).C \notin \mathcal{L}(y)$,
then	$\mathcal{L}(y) := \mathcal{L}(y) \cup \{\forall \mathcal{B}(q).C\}$
$\forall_3$-rule: if	$\forall \mathcal{B}.C \in \mathcal{L}(x)$, $\varepsilon \in L(\mathcal{B})$, $\mathsf{tv}(C) \cap \mathbf{N}_s = \emptyset$, x is not indirectly blocked, and $C \notin \mathcal{L}(x)$,
then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{C\}$
$\geqslant$-rule: if	1. $(\geqslant nR.C) \in \mathcal{L}(x)$, $\mathsf{tv}(C) \cap \mathbf{N}_s = \emptyset$, 2. x is not blocked, and there are not n safe R-neighbors $y_1, \ldots, y_n$ of x with $C \in \mathcal{L}(y_i)$ and $y_i \dot{\neq} y_j$ for $1 \leqslant i, j \leqslant n$, $i \neq j$,
then	create n new nodes $y_1, \ldots, y_n$ such that $\mathcal{L}(\langle x, y_i\rangle) := \{R\}$, $\mathcal{L}(y_i) = \{C, \top\}$ and $y_i \dot{\neq} y_j$ for $1 \leqslant i, j \leqslant n$, $i \neq j$
ch-rule: if	$(\leqslant nR.C) \in \mathcal{L}(x)$, $\mathsf{tv}(C) \cap \mathbf{N}_s = \emptyset$, x is not indirectly blocked, and x has an R-neighbor y with $\{C, \mathsf{NNF}(\neg C)\} \cap \mathcal{L}(y) = \emptyset$,
then	$\mathcal{L}(y) := \mathcal{L}(y) \cup \{D\}$ for some $D \in \{C, \mathsf{NNF}(\neg C)\}$
$\leqslant$-rule: if	$(\leqslant nR.C) \in \mathcal{L}(x)$, x is not indirectly blocked, $\mathsf{tv}(C) \cap \mathbf{N}_s = \emptyset$, $\#R^{\mathbf{G}}(x, C) > n$, and x has two R-neighbors y, z such that $C \in \mathcal{L}(y) \cap \mathcal{L}(z)$, and not $y \dot{\neq} z$,
then	1. if y is a nominal node, $\textsc{Merge}(z, y)$ 2. else if z is a nominal node or an ancestor of y, then $\textsc{Merge}(y, z)$ 3. else $\textsc{Merge}(z, y)$
o-rule: if	there are 2 nodes x, y with $o \in \mathcal{L}(x) \cap \mathcal{L}(y)$ for some $o \in \mathbf{N}_o$, and $x \dot{\neq} y$ does not hold,
then	$\textsc{Merge}(x, y)$

Figure 4: Expansion rules for $\mathcal{SROIQV}$ (part 1)

NN-rule: if	1. $(\leqslant nR.C) \in \mathcal{L}(x)$, $\mathsf{tv}(C) \cap \mathbf{N}_s = \emptyset$, 2. x is a nominal node, x has a blockable R-neighbor y with $C \in \mathcal{L}(y)$ and x is a successor of y, 3. there is no k with $1 \leqslant k \leqslant n$, $(\leqslant kR.C) \in \mathcal{L}(x)$, and x has k nominal R-neighbors $z_1, \ldots, z_k$ with $C \in \mathcal{L}(z_i)$ for $1 \leqslant i \leqslant k$ and $z_i \dot{\neq} z_j$ for $1 \leqslant i < j \leqslant k$,
then	1. guess k with $1 \leqslant k \leqslant n$, and set $\mathcal{L}(x) := \mathcal{L}(x) \cup \{\leqslant kR.C\}$ 2. create k new nodes $y_1, \ldots, y_k$ with $\mathcal{L}(\langle x, y_i\rangle) = \{R\}$, $\mathcal{L}(y_i) = \{C_i, o_i, \top\}$ for $1 \leqslant i \leqslant k$ such that $o_1, \ldots, o_k \in \mathbf{N}_a$ are unique and new in $\mathbf{G}$ 3. set $y_i \dot{\neq} y_j$ for $1 \leqslant i < j \leqslant k$
G1-rule: if	1. $D \in \mathcal{L}(x)$ where D is of the form $\{v\}$ or $\neg\{v\}$ or $C_1 \sqcap C_2$ or $C_1 \sqcup C_2$ where $v \in \mathsf{N}_V$, 2. $\mathsf{tv}(D) \cap \mathbf{N}_s \neq \emptyset$, 3. $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D) \notin \mathcal{L}(x)$ for some v-assignment $\mathcal{Z}$ where $\Gamma = \mathsf{Var}(\mathsf{tv}(D) \cap \mathbf{N}_s)$
then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \mid C \in \mathcal{L}(x)\}$
G2-rule: if	1. $D \in \mathcal{L}(x)$ where D is of the form $\exists R.E$, or $(\geqslant nR.E)$, or $(\leqslant nR.E)$, 2. $\mathsf{tv}(E) \cap \mathbf{N}_s \neq \emptyset$, 3. $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D) \notin \mathcal{L}(x)$ for some v-assignment $\mathcal{Z}$ where $\Gamma = \mathsf{Var}(\mathsf{tv}(E) \cap \mathbf{N}_s)$
then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \mid C \in \mathcal{L}(x)\}$
G3-rule: if	1. $\forall \mathcal{B}(p).E \in \mathcal{L}(x)$, $\varepsilon \in L(\mathcal{B}(p))$, 2. $\mathsf{tv}(E) \cap \mathbf{N}_s \neq \emptyset$, 3. $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\forall \mathcal{B}(p).E) \notin \mathcal{L}(x)$ for some v-assignment $\mathcal{Z}$ where $\Gamma = \mathsf{Var}(\mathsf{tv}(E) \cap \mathbf{N}_s)$
then	$\mathcal{L}(x) := \mathcal{L}(x) \cup \{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \mid C \in \mathcal{L}(x)\}$

Figure 5: Expansion rules for $\mathcal{SROIQV}$ (part 2)

Note that our algorithm indeed *delays* grounding. This can be seen, e.g., by considering a label $\exists R_1.\exists S_1.\{x\} \sqcap \exists R_2.\exists S_2.\{x\} \in \mathcal{L}z$. In this case, it is possible to first apply the $\sqcap$-rule and then the $\exists$-rule to each conjunct, so that the nominal schema $\{x\}$ in fact ends up in the labels of two different nodes of the graph. While at first sight this seems to destroy the variable binding, it in fact doesn't—because each nominal schema must be grounded in all possible ways before a complete clash-free graph can be obtained, i.e., one of the choices for grounding will identify the two occurrences of this nominal schema.

An Example

To describe how the algorithm works, we extend the example from Figure 1 into the knowledge base KB in Figure 6. Given KB, we ask whether it entails the existence of a conflicting review assignment, i.e., whether the concept $D := \forall$hasConflictingAssignedPaper.$\bot$ is unsatisfiable w.r.t. KB.

The answer must be yes since a_1 and a_{1000} co-author p_0, but a_1 has review assignment on p_{999} whose authors include a_{1000}. To run the algorithm, we first internalize KB by transforming each axiom $C_1 \sqsubseteq C_2$ into a concept $\mathsf{NNF}(\neg C \sqcup D)$, take the conjunction of all of such concepts and combine it with D as described by the algorithm. This knowledge base has 2001 individual names and three nominal schema $\{x\}$, $\{y\}$ and $\{z\}$. Note that if we solve this entailment by grounding the first axiom up front, we would have more than 8×10^9 new axioms.

On the other hand, our algorithm can discover the clash without necessarily grounding everything up front. Initially, an initial completion graph is constructed with nodes, say,

$$\begin{aligned}
&\exists \text{hasReviewAssignment}.((\{x\} \sqcap \exists \text{hasAuthor}.\{y\}) \sqcap (\{x\} \sqcap \exists \text{atVenue}.\{z\}))\\
&\qquad\qquad \sqcap \exists \text{hasSubmittedPaper}.(\exists \text{hasAuthor}.\{y\} \sqcap \exists \text{atVenue}.\{z\}) \sqsubseteq \exists \text{hasConflictingAssignedPaper}.\{x\}\\
&\qquad \{p_0\} \sqsubseteq \exists \text{hasAuthor}.\{a_{1000}\} \sqcap \exists \text{hasAuthor}.\{a_1\}\\
&\qquad \{p_i\} \sqsubseteq \exists \text{hasAuthor}.\{a_i\} \sqcap \exists \text{hasAuthor}.\{a_{i+1}\}\\
&\qquad \{a_i\} \sqsubseteq \exists \text{hasSubmittedPaper}.\{p_{i-1}\} \sqcap \exists \text{hasSubmittedPaper}.\{p_i\}\\
&\qquad \{a_{1000}\} \sqsubseteq \exists \text{hasSubmittedPaper}.\{p_{999}\} \sqcap \exists \text{hasSubmittedPaper}.\{p_0\}\\
&\qquad \{p_j\} \sqsubseteq \exists \text{AtVenue}.\{\text{ISWC}\}\\
&\qquad \{a_k\} \sqsubseteq \exists \text{hasReviewAssignment}.\{p_{k-4}\} \sqcap \exists \text{hasReviewAssignment}.\{p_{k-3}\}\\
&\qquad \{a_1\} \sqsubseteq \exists \text{hasReviewAssignment}.\{p_{999}\} \sqcap \exists \text{hasReviewAssignment}.\{p_{998}\}
\end{aligned}$$

Figure 6: Example for delayed grounding. $i = 1, \dots, 999$, $j = 0, \dots, 999$, $k = 4, \dots, 1000$.

r_0, $a_0, \dots, a_{999}$, and $p_0, \dots, p_{999}$, and $iswc$, with r_0 is where we place the initial concept and internalized KB. We can quickly propagate the internalized GCIs into r_0, p_{999}, and a_{1000}. Because r_0 contains no nominal in its label, r_0 will immediately receive the RHSs of all GCIs in KB, except the first GCI. Next, after a few applications of the $\sqcap$-rule, we apply the $\exists$-rule and the o-rule to r_0, a_{1000} and p_{999}. We obtain, without applying the grounding rule:

- p_0 is a successor of both r_0 and a_{1000} through the hasSubmittedPaper role.
- p_{999} is a successor of r_0 through the hasReviewAssignment role.
- p_{999} is a successor of a_{1000} through the hasSubmittedPaper role.
- p_0 and p_{999} has $iswc$ as their successor through the atVenue role.
- a_{1000} is a successor of p_{999} through the hasAuthor role.

Next, we apply the $\sqcup$-rule on r_0 to choose one of the disjuncts from $\mathsf{NNF}(\neg C_1 \sqcup D_1)$ where $C_1 \sqsubseteq D_1$ is the first axiom in KB (the only one containing occurrences of nominal schemas). Suppose the first disjunct is chosen on r_0. Next, apply $\forall$-rules and the GR3-rule to eventually obtain a clash when $\neg\{x\}$ is grounded to $\neg\{p_{999}\}$ on p_{999}. If we choose to propagate $\neg\{y\}$ through the hasAuthor role, again we get a clash when grounding $\neg\{y\}$ to $\neg\{a_{1000}\}$ on a_{1000}. Similarly, a clash also occurs when we propagate $\neg\{z\}$ through the atVenue role.

Next, suppose the second disjunct is chosen on r_0. Propagating through the hasSubmittedRole means that p_0 must contain the label $\forall$hasAuthor.$\neg\{y\} \sqcup \forall$atVenue.$\neg\{z\}$. Similar to p_{999}, any choice of disjunct here will lead to a clash. So we can only now select the third disjunct on r_0: $\exists$hasConflictingAssignedPaper.$\{x\}$. Here, grounding x to p_{999} and applying the $\exists$-rule followed by the o-rule will connect r_0 to p_{999} through the hasConflictingAssignedPaper role. But then, r_0 also contains $\forall$hasConflictingAssignedPaper.$\bot$ in its label. Hence, applying the $\forall$-rules will propagate $\bot$ to p_{999} leading to a clash.

So, by only grounding x to p_{999}, y to a_{1000} and z to ISWC, we can discover the clashes quickly and derive the unsatisfiability of $\forall$hasConflictingAssignedPaper.$\bot$. By delaying grounding and then performing it selectively, we can avoid the combinatorial explosion which is unavoidable when grounding is done up front. Obviously, this does not mean that combinatorial explosion can always be avoided. On the other hand, this idea can lead to more practical heuristics that enable efficient implementations for reasoning with nominal schemas.

Correctness Proofs

We now proceed with showing termination, soundness, and completeness of our algorithm.

Lemma 11 (Termination)**.** *Given a $\mathcal{SROIQV}$ concept C_0 in NNF and a reduced RBox $\mathcal{R}$, the tableaux algorithm for C_0 and $\mathcal{R}$ terminates.*

Proof. Note that each application of G1-, G2-, and G3-rules adds new partially grounded $\mathcal{SROIQV}$ concepts to concept labeling of a node in the completion graph. Thus, the number of those rule applications is bounded by the number of partially grounded $\mathcal{SROIQV}$ concept that can be generated from $\mathsf{fcl}(C_0, \mathcal{R})$ which is finite due to the fact the number of all possible v-assignments is finite. The termination would then follow from this, together with the termination of $\mathcal{SROIQ}$ tableau algorithm (Horrocks, Kutz, and Sattler 2006). $\square$

Let C_0 be a $\mathcal{SROIQV}$ concept, $\mathcal{R}$ be a regular RBox, $\mathbf{N}_s$ (resp. $\mathbf{N}_o$) be the set of all nominal schemas (resp. nominals) occurring in C_0, and $\mathbf{N}_{slf}$ be the set of concepts of the form $\exists R.\mathsf{Self}$ occurring in C_0. Let Φ be the (finite) set of all v-assignments from N_V to N_I. A *tableau* for C_0 w.r.t. $\mathcal{R}$ is a triple $T = (\mathbf{S}, \mathcal{L}, \mathcal{E})$ where $\mathbf{S}$ is a nonempty set, $\mathcal{L}$ maps each $s \in \mathbf{S}$ to $\mathcal{L}(s) \subseteq \mathsf{fcl}(C_0, \mathcal{R}) \cup \mathsf{gcl}(C_0, \mathcal{R})$, the mapping $\mathcal{E}$ maps each role $R \in \mathsf{rol}(C_0, \mathcal{R})$ to $\mathcal{E}(R) \subseteq \mathbf{S} \times \mathbf{S}$, and $C_0 \in \mathcal{L}(s)$ for some $s \in \mathbf{S}$. In addition, for every $s, t \in \mathbf{S}$, $C, D \in \mathsf{fcl}(C_0, \mathcal{R})$, $R, S \in \mathsf{rol}(C_0, \mathcal{R})$ and

$$\begin{aligned}
R^T(s, C) = \{s' \in \mathbf{S} \mid {} & \text{there is an } R' \in L(\mathcal{B}_R) \text{ such that}\\
& \langle s, s' \rangle \in \mathcal{E}(R') \text{ and } C \in \mathcal{L}(s')\}
\end{aligned}$$

the tableau T also satisfies the properties below.

(P1) $\langle s, t \rangle \in \mathcal{E}(R)$ iff $\langle t, s \rangle \in \mathcal{E}(\mathsf{Inv}(R))$

(P2) If $\mathsf{Dis}(R, S) \in \mathcal{R}$, then $\mathcal{E}(R) \cap \mathcal{E}(S) = \emptyset$.

(P3) If $\langle s, t \rangle \in \mathcal{E}(R)$ and $R \mathrel{\underline{\overset{*}{\sqsubset}}} S$, then $\langle s, t \rangle \in \mathcal{E}(S)$.

(P4) For every $o \in \mathbf{N}_o$, $o \in \mathcal{L}(s')$ for some $s' \in \mathbf{S}$.

(P5) If $o \in \mathcal{L}(s) \cap \mathcal{L}(t)$ for some $o \in \mathbf{N}$, then $s = t$.

(P6) For every $v \in \mathbf{N}_s$, if $v \in \mathcal{L}(s)$, then $o \in \mathcal{L}(s)$ for every $o \in \mathbf{N}_o$.

(P7) For $C \in \mathsf{N}_C \cup \mathbf{N}_o \cup \mathbf{N}_{slf}$, if $C \in \mathcal{L}(s)$,
then $\neg C \notin \mathcal{L}(s)$.

(P8) For every $v \in \mathbf{N}_s$, if $v \in \mathcal{L}(s)$, then for every v-assignment $\mathcal{Z} \in \Phi$, $\mathsf{gr}_{\mathcal{Z}}(\neg v) \notin \mathcal{L}(s)$.

(P9) $\top \in \mathcal{L}(s)$ and $\bot \notin \mathcal{L}(s)$.

(P10) If $\exists R.\mathsf{Self} \in \mathcal{L}(s)$, then $\langle s, s\rangle \in \mathcal{E}(R)$.

(P11) If $\neg\exists R.\mathsf{Self} \in \mathcal{L}(s)$, then $\langle s, s\rangle \notin \mathcal{E}(R)$.

(P12) If $C \sqcap D \in \mathcal{L}(s)$, then for every v-assignment $\mathcal{Z} \in \Phi$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C \sqcap D) \cap \mathbf{N}_s)$, it holds that $\{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C \sqcap D), \mathsf{gr}_{\mathcal{Z}_\Gamma}(C), \mathsf{gr}_{\mathcal{Z}_\Gamma}(D)\} \subseteq \mathcal{L}(s)$.

(P13) If $C \sqcup D \in \mathcal{L}(s)$, then for every v-assignment $\mathcal{Z} \in \Phi$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C \sqcup D) \cap \mathbf{N}_s)$, it holds that $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C \sqcup D) \in \mathcal{L}(s)$ and $\{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C), \mathsf{gr}_{\mathcal{Z}_\Gamma}(D)\} \cap \mathcal{L}(s) \neq \emptyset$.

(P14) If $\forall R.C \in \mathcal{L}(s)$, then $\forall \mathcal{B}_R.C \in \mathcal{L}(s)$.

(P15) If $\forall \mathcal{B}(p).C \in \mathcal{L}(s)$, $\langle s, t\rangle \in \mathcal{E}(R)$,
and $p \xrightarrow{R} q \in \mathcal{B}(p)$, then $\forall \mathcal{B}(q).C \in \mathcal{L}(t)$.

(P16) If $\forall \mathcal{B}.C \in \mathcal{L}(s)$, and $\varepsilon \in L(\mathcal{B})$, then for every $\mathcal{Z} \in \Phi$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s)$,
$\{\mathsf{gr}_{\mathcal{Z}_\Gamma}(\forall \mathcal{B}.C), \mathsf{gr}_{\mathcal{Z}_\Gamma}(C)\} \subseteq \mathcal{L}(s)$.

(P17) If $\exists R.C \in \mathcal{L}(s)$, then for every $\mathcal{Z} \in \Phi$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s)$, $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\exists R.C) \in \mathcal{L}(s)$ and there is an $s' \in \mathbf{S}$ with $\langle s, s'\rangle \in \mathcal{E}(R)$ and $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \in \mathcal{L}(s')$.

(P18) If $(\geqslant nR.C) \in \mathcal{L}(s)$, then for every v-assignment $\mathcal{Z} \in \Phi$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s)$, $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\geqslant nR.C) \in \mathcal{L}(s)$ and $\#R^T(s, \mathsf{gr}_{\mathcal{Z}_\Gamma}(C)) \geqslant n$.

(P19) If $(\leqslant nR.C) \in \mathcal{L}(s)$, then for every v-assignment $\mathcal{Z} \in \Phi$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s)$, $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\leqslant nR.C) \in \mathcal{L}(s)$ and $\#R^T(s, \mathsf{gr}_{\mathcal{Z}_\Gamma}(C)) \leqslant n$.

(P20) If $(\leqslant nR.C) \in \mathcal{L}(s)$, and $\langle s, t\rangle \in \mathcal{E}(R)$, then for every v-assignment $\mathcal{Z} \in \Phi$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s)$, either $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \in \mathcal{L}(t)$ or $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\mathsf{NNF}(\neg C)) \in \mathcal{L}(t)$, but not both.

Lemma 12. *A $\mathcal{SROIQV}$ concept C_0 is satisfiable w.r.t. a reduced RBox $\mathcal{R}$ iff there exists a tableau for C_0 w.r.t. $\mathcal{R}$.*

Proof. We begin with the "if" direction. Let $T = (\mathbf{S}, \mathcal{L}, \mathcal{E})$ be a tableau for C_0 w.r.t. $\mathcal{R}$. We define an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where $\Delta^\mathcal{I} = \mathbf{S} \neq \emptyset$. Also, for concept names A, $A^\mathcal{I} = \{s \in \Delta^\mathcal{I} \mid A \in \mathcal{L}(s)\}$, and for individual names a, $a^\mathcal{I} = s$ for some $s \in \mathbf{S}$ with $\{a\} \in \mathcal{L}(s)$. Notice that due to (P4) and (P5), for each nominal $o \in \mathbf{N}_o$, there is exactly one (not necessarily unique) element $s \in \Delta^\mathcal{I}$ for which $o \in \mathcal{L}(s)$. For role names $R \in \mathsf{rol}(C_0, \mathcal{R})$, we define:

$$\begin{aligned} R^\mathcal{I} = \{\langle t_0, t_n\rangle \in \mathbf{S}^2 \mid {} & \text{there are } t_1, \ldots, t_{n-1} \in \mathbf{S} \text{ with} \\ & \langle t_i, t_{i+1}\rangle \in \mathcal{E}(R_{i+1}) \text{ for } 0 \leqslant i < n, \\ & \text{and } R_1 \cdots R_n \in L(\mathcal{B}_R)\} \end{aligned}$$

Due to the semantics, for $A \in \mathsf{N}_C$, $a \in \mathsf{N}_I$ and role names $R \in \mathsf{N}_R$, we have that $A^{\mathcal{I},\mathcal{Z}} = A^\mathcal{I}$, $a^{\mathcal{I},\mathcal{Z}} = a^\mathcal{I}$ and $R^{\mathcal{I},\mathcal{Z}} = R^\mathcal{I}$ for every v-assignment $\mathcal{Z}$. Furthermore, for every role name R, $(R^-)^\mathcal{I} = \{\langle s, t\rangle \mid \langle t, s\rangle \in R^\mathcal{I}\}$ because of (P1) and Lemma 9.3.

We next show that $\mathcal{I} \models \mathcal{R}$. Here, v-assignments does not affect the fact whether or not $\mathcal{I} \models \mathcal{R}$. First, for any simple role R, $R^\mathcal{I} = \mathcal{E}(R)$, due to definition of $\mathcal{I}$, Lemma 9.2, (P1), and (P3). Moreover, for any assertion $\mathsf{Dis}(R, S) \in \mathcal{R}$ with R, S simple, $R^\mathcal{I} \cap S^\mathcal{I} = \emptyset$ due to (P2). It follows that $\mathcal{I} \models \mathsf{Dis}(R, S)$.

Next, we show that $\mathcal{I} \models w \sqsubseteq R$ for each $w \sqsubseteq R \in \mathcal{R}$. Using Theorem 8, it suffices to show that for every (possibly inverse) role R and $w \in L(\mathcal{B}_R)$, $\langle s, t\rangle \in w^\mathcal{I}$ implies $\langle s, t\rangle \in R^\mathcal{I}$. Assume R is a (possibly inverse) role, $w = R_1 \cdots R_n \in L(\mathcal{B}_R)$ and $\langle s, t\rangle \in w^\mathcal{I}$ for some $s, t \in \Delta^\mathcal{I}$. Since $\langle s, t\rangle \in w^\mathcal{I}$, there are $t_0, t_1, \ldots, t_{n-1}, t_n \in \mathbf{S}_\alpha$ such that $s = t_0$, $t = t_n$, and $\langle t_{i-1}, t_i\rangle \in R_i^\mathcal{I}$, $0 < i \leqslant n$. In order to derive $\langle s, t\rangle \in R^\mathcal{I}$, the following needs to be shown: there are $s_0, \ldots, s_{n'} \in \mathbf{S}_\alpha$ such that $s = s_0$, $t = s_{n'}$, $\langle s_{j-1}, s_j\rangle \in \mathcal{E}(S_j)$ for some role S_j, $0 < j \leqslant n'$, and $S_1 \cdots S_{n'} \in L(\mathcal{B}_R)$ (*).

If $\langle t_{i-1}, t_i\rangle \in \mathcal{E}(R_i)$ for all $0 < i \leqslant n$, then $n' = n$, $s_j = t_j$ for $0 \leqslant j \leqslant n$ and we are done. Otherwise, $\langle t_{i-1}, t_i\rangle \notin \mathcal{E}(R_i)$ for some i, $0 < i \leqslant n$. Since $\langle t_{i-1}, t_i\rangle \in R_i^\mathcal{I}$, there are $y_0^{(i)}, \ldots, y_{n_i}^{(i)} \in \mathbf{S}_\alpha$ and roles $T_1^{(i)}, \ldots, T_{n_i}^{(i)}$ such that $y_0^{(i)} = t_{i-1}$, $y_{n_i}^{(i)} = t_i$, $\langle y_{j-1}^{(i)}, y_j^{(i)}\rangle \in \mathcal{E}(T_j^{(i)})$ for $0 < j \leqslant n_i$, and $T_1^{(i)} \cdots T_{n_i}^{(i)} \in L(\mathcal{B}_{R_i})$. Construction of $\mathcal{B}_R$, as given in (Horrocks and Sattler 2004), implies that $w' \in L(\mathcal{B}_R)$ where w' is obtained from w by replacing R_i with the word $T_1^{(i)} \cdots T_{n_i}^{(i)}$. Repeatedly doing this replacement for every i, $0 < i \leqslant n$, for which $\langle t_{i-1}, t_i\rangle \notin \mathcal{E}(R_i)$ will yield in the existence of $s_0, \ldots, s_{n'}$ and roles $S_1, \ldots, S_{n'}$ satisfying (*). Hence, $\langle s, t\rangle \in R^\mathcal{I}$.

Next, we show that $\mathcal{I} \models C_0$. Since $C_0 \in \mathcal{L}(s)$ for some $s \in \mathbf{S}$, it suffices to show, for each $C \in \mathsf{fcl}(C_0, \mathcal{R})$ and $s \in \mathbf{S}$, that $C \in \mathcal{L}(s)$ implies $s \in C^{\mathcal{I},\mathcal{Z}}$ for every v-assignment $\mathcal{Z}$. This claim is shown by induction on the structure of C. The base case for concept names and nominals, as well as the case for concept complement, are trivial. The induction is similar to the one from (Horrocks, Kutz, and Sattler 2006).

- Let $v \in \mathcal{L}(s)$ for some $v \in \mathbf{N}_s$. Then (P6) implies that $o \in \mathcal{L}(s)$ for every $o \in \mathbf{N}_o$. Thus, $s \in v^{\mathcal{I},\mathcal{Z}}$ for every v-assignment $\mathcal{Z}$ because given an assignment $\mathcal{Z}$, $v^{\mathcal{I},\mathcal{Z}} = o$ for some $o \in \mathbf{N}_o$, and hence the induction applies.
- Let $\neg v \in \mathcal{L}(s)$ for some $v \in \mathbf{N}_s$. First, we have that $o \notin \mathcal{L}(s)$ for every $o \in \mathbf{N}_o$, by (P8) and definition of v-assignment. Hence, by induction, $s \notin o^{\mathcal{I},\mathcal{Z}}$ for every $o \in \mathbf{N}_o$. It thus follows that, for every v-assignment $\mathcal{Z}$, $s \notin v^{\mathcal{I},\mathcal{Z}}$, i.e., $s \in (\neg v)^{\mathcal{I},\mathcal{Z}}$.
- Let $C \sqcap D \in \mathcal{L}(s)$. Let $\mathcal{Z}$ be any v-assignment and $\Gamma = \mathsf{Var}(\mathsf{tv}(C \sqcap D) \cap \mathbf{N}_s)$. Due to (P12), $\{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C), \mathsf{gr}_{\mathcal{Z}_\Gamma}(D)\} \subseteq \mathcal{L}(s)$. By induction, $s \in (\mathsf{gr}_{\mathcal{Z}_\Gamma}(C))^{\mathcal{I},\mathcal{Z}}$ and $s \in (\mathsf{gr}_{\mathcal{Z}_\Gamma}(D))^{\mathcal{I},\mathcal{Z}}$. Since $\mathcal{Z}_\Gamma$ is a (possibly empty) Γ-restriction of $\mathcal{Z}$, by Lemma 7, $(\mathsf{gr}_{\mathcal{Z}_\Gamma}(C))^{\mathcal{I},\mathcal{Z}} = (\mathsf{gr}_{\mathcal{Z}}(C))^{\mathcal{I},\mathcal{Z}} = C^{\mathcal{I},\mathcal{Z}}$. This similarly holds for $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D)$. Hence, $s \in C^{\mathcal{I},\mathcal{Z}} \cap D^{\mathcal{I},\mathcal{Z}}$, i.e., $s \in (C \sqcap D)^{\mathcal{I},\mathcal{Z}}$ as desired.
- Let $C \sqcup D \in \mathcal{L}(s)$. Let $\mathcal{Z}$ be any v-assignment and $\Gamma = \mathsf{Var}(\mathsf{tv}(C \sqcup D) \cap \mathbf{N}_s)$. Either $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \in \mathcal{L}(s)$ or $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D) \in \mathcal{L}(s)$ due to (P13). By induction and Lemma 7, either $s \in (\mathsf{gr}_{\mathcal{Z}_\Gamma}(C))^{\mathcal{I},\mathcal{Z}} = C^{\mathcal{I},\mathcal{Z}}$ or $(\mathsf{gr}_{\mathcal{Z}_\Gamma}(D))^{\mathcal{I},\mathcal{Z}} = D^{\mathcal{I},\mathcal{Z}}$. So, $s \in C^{\mathcal{I},\mathcal{Z}} \cup D^{\mathcal{I},\mathcal{Z}}$, i.e., $s \in (C \sqcup D)^{\mathcal{I},\mathcal{Z}}$.
- If $\forall R.D \in \mathcal{L}(s)$, then by (P14), $\forall \mathcal{B}_R.D \in \mathcal{L}(s)$. Let $\mathcal{Z}$

be any v-assignment. Assume that $\langle s,t\rangle \in R^{\mathcal{I},\mathcal{Z}}$ for some $t \in \mathbf{S}$. By definition of $R^{\mathcal{I},\mathcal{Z}}$, there exists $t_0,\dots,t_n \in \mathbf{S}_\alpha$ and $R_1,\dots,R_n$ such that $s=t_0$, $t=t_n$, $\langle t_{i-1},t_i\rangle \in \mathcal{E}(R_i)$ for every $0<i\leqslant n$, and $R_1\cdots R_n \in L(\mathcal{B}_R)$. Applying (P15) n times yields $\forall\mathcal{B}_R(q).D \in \mathcal{L}(t)$ and $\varepsilon \in L(\mathcal{B}_R(q))$. (P16) implies that $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D) \in \mathcal{L}(t)$ where $\Gamma = \mathsf{Var}(\mathsf{tv}(D)\cap \mathbf{N}_s)$. By induction and Lemma 7, $t \in \mathsf{gr}_{\mathcal{Z}_\Gamma}(D)^{\mathcal{I},\mathcal{Z}} = D^{\mathcal{I},\mathcal{Z}}$. Hence, $s \in (\forall R.D)^{\mathcal{I},\mathcal{Z}}$.

- Let $\exists R.\mathsf{Self} \in \mathcal{L}(s)$. Let $\mathcal{Z}$ be any v-assignment. It follows from (P10) that $\langle s,s\rangle \in \mathcal{E}(R)$. Due to $R \in \mathcal{B}_R$ and definition of $R^{\mathcal{I}}$, $\langle s,s\rangle \in R^{\mathcal{I}} = R^{\mathcal{I},\mathcal{Z}}$. Hence, $s \in (\exists R.\mathsf{Self})^{\mathcal{I},\mathcal{Z}}$.
- Let $\neg\exists R.\mathsf{Self} \in \mathcal{L}(s)$. Let $\mathcal{Z}$ be any v-assignment. Then (P11) and the fact that R is simple imply that $\langle s,s\rangle \notin R^{\mathcal{I}} = R^{\mathcal{I},\mathcal{Z}}$. Hence, $s \in (\neg\exists R.\mathsf{Self})^{\mathcal{I},\mathcal{Z}}$.
- Let $\exists R.D \in \mathcal{L}(s)$. Let $\mathcal{Z}$ be any v-assignment. By (P17), there is some t such that $\langle s,t\rangle \in \mathcal{E}(R)$ and $t \in (\mathsf{gr}_{\mathcal{Z}_\Gamma}(D))^{\mathcal{I},\mathcal{Z}}$. Definition of $R^{\mathcal{I}}$, the induction and Lemma 7 imply that $\langle s,t\rangle \in R^{\mathcal{I},\mathcal{Z}}$ and $t \in D^{\mathcal{I},\mathcal{Z}}$. So, $s \in (\exists R.D)^{\mathcal{I},\mathcal{Z}}$ as required.
- Let $(\geqslant nR.D) \in \mathcal{L}(s)$. Let $\mathcal{Z}$ be any v-assignment and $\Gamma = \mathsf{Var}(\mathsf{tv}(D)\cap\mathbf{N}_s)$. It follows from (P18) that $\#R^{T}(s,\mathsf{gr}_{\mathcal{Z}_\Gamma}(D)) \geqslant n$. So, there are n elements $t_1,\dots,t_n \in \mathbf{S}$ such that $\langle s,t_i\rangle \in \mathcal{E}(R)$ and $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D) \in \mathcal{L}(t_i)$. By induction, definition of $R^{\mathcal{I}}$ and Lemma 7, we obtain that $\langle s,t_i\rangle \in R^{\mathcal{I},\mathcal{Z}}$ and $t_i \in \mathsf{gr}_{\mathcal{Z}_\Gamma}(D)^{\mathcal{I},\mathcal{Z}} = D^{\mathcal{I},\mathcal{Z}}$ for $1 \leqslant i \leqslant n$. Hence, $s \in (\geqslant nR.D)^{\mathcal{I},\mathcal{Z}}$.
- Let $(\leqslant nR.D) \in \mathcal{L}(s)$. Let $\mathcal{Z}$ be any v-assignment and $\Gamma = \mathsf{Var}(\mathsf{tv}(D)\cap\mathbf{N}_s)$. Since R is simple, $R^{\mathcal{I},\mathcal{Z}} = R^{\mathcal{I}} = \mathcal{E}(R)$. By (P20), for every t, if $\langle s,t\rangle \in R^{\mathcal{I}}$, then $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D) \in \mathcal{L}(t)$ or $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\mathsf{NNF}(\neg D)) \in \mathcal{L}(t)$. It follows from (P19) that $\#R^{T}(s,\mathsf{gr}_{\mathcal{Z}_\Gamma}(D)) \leqslant n$. Hence there are at most n elements t for which $\langle s,t\rangle \in \mathcal{E}(R)$ and $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D) \in \mathcal{L}(t)$. By induction and Lemma 7, we have for those t's that $t \in D^{\mathcal{I},\mathcal{Z}}$. So, $s \in (\leqslant nR.D)^{\mathcal{I},\mathcal{Z}}$.

For the "only if" direction, let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model of C_0 and $\mathcal{R}$, i.e., $C_0^{\mathcal{I},\mathcal{Z}} \neq \emptyset$ and $\mathcal{I},\mathcal{Z} \models \mathcal{R}$ for every v-assignment $\mathcal{Z}$. Suppose Φ is the set of all such $\mathcal{Z}$ which is finite and always exists for a given C_0. Note that $R^{\mathcal{I},\mathcal{Z}} = R^{\mathcal{I}}$ for every $\mathcal{Z} \in \Phi$. A tableau $T = (\mathbf{S},\mathcal{L},\mathcal{E})$ for C_0 and $\mathcal{R}$ can be defined as follows. We set $\mathbf{S} := \Delta^{\mathcal{I}}$. For every $R \in \mathsf{rol}(C_0,\mathcal{R})$, $\mathcal{E}(R) := R^{\mathcal{I}}$.

For every $s \in \mathbf{S}$, we define $\mathcal{L}(s)$ as follows. First, we define $\mathcal{L}'(s)$ as in Figure 7. Then we define $\mathcal{L}'(s)$ from $\mathcal{L}(s)$ as the smallest set that satisfies the following:

- $\mathcal{L}'(s) \subseteq \mathcal{L}(s)$ for each $s \in \mathbf{S}$;
- for each $s \in \mathbf{S}$ and each $\mathcal{Z} \in \Phi$, if $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C \sqcap D) \in \mathcal{L}(s)$, then $\{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C), \mathsf{gr}_{\mathcal{Z}_\Gamma}(D)\} \subseteq \mathcal{L}(s)$ where $\Gamma = \mathsf{Var}(\mathsf{tv}(C \sqcap D)\cap\mathbf{N}_s)$;
- for each $s \in \mathbf{S}$ and each $\mathcal{Z} \in \Phi$, if $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C \sqcup D) \in \mathcal{L}(s)$ and $s \in C^{\mathcal{I},\mathcal{Z}}$, then $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \in \mathcal{L}(s)$ where $\Gamma = \mathsf{Var}(\mathsf{tv}(C \sqcup D)\cap\mathbf{N}_s)$;
- for each $s \in \mathbf{S}$ and each $\mathcal{Z} \in \Phi$, if $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C \sqcup D) \in \mathcal{L}(s)$ and $s \in D^{\mathcal{I},\mathcal{Z}}$, then $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D) \in \mathcal{L}(s)$ where $\Gamma = \mathsf{Var}(\mathsf{tv}(C \sqcup D)\cap\mathbf{N}_s)$;
- for all $\mathcal{Z} \in \Phi$, if $s,t \in \mathbf{S}$ satisfy all of the following:
  (i) $\langle s,t\rangle \in R^{\mathcal{I}}$, and
  (ii) $s \in C^{\mathcal{I},\mathcal{Z}}$ where $C \in \mathsf{fcl}(C_0,\mathcal{R})$ is of the form $\exists R.D$, or $(\geqslant nR.D)$, or $(\leqslant nR.D)$,

  then: with $\Gamma = \mathsf{Var}(\mathsf{tv}(D)\cap\mathbf{N}_s)$, we have
  - if $t \in D^{\mathcal{I},\mathcal{Z}}$, then $\mathsf{gr}_{\mathcal{Z}_\Gamma}(D) \in \mathcal{L}(t)$,
  - otherwise, $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\mathsf{NNF}(\neg D)) \in \mathcal{L}(t)$.

$$
\begin{aligned}
\mathcal{L}'(s) := {} & \{C \in \mathsf{fcl}(C_0,\mathcal{R}) \mid s \in C^{\mathcal{I},\mathcal{Z}} \text{ for all } \mathcal{Z} \in \Phi\} \\
& \cup \{\forall\mathcal{B}_R.C \mid \forall R.C \in \mathsf{fcl}(C_0,\mathcal{R}) \\
& \qquad \text{and } s \in (\forall R.C)^{\mathcal{I},\mathcal{Z}} \text{ for all } \mathcal{Z} \in \Phi\} \\
& \cup \{\forall\mathcal{B}(q).C \in \mathsf{fcl}(C_0,\mathcal{R}) \mid \\
& \qquad \text{for all } R_1\cdots R_n \in L(\mathcal{B}(q)), \text{ for all } \mathcal{Z} \in \Phi, \\
& \qquad s \in (\forall R_1.\forall R_2.\cdots.\forall R_n.C)^{\mathcal{I},\mathcal{Z}}, \text{ and} \\
& \qquad \text{if } \varepsilon \in L(\mathcal{B}_R(q)) \text{ then } s \in C^{\mathcal{I},\mathcal{Z}}\} \\
& \cup \{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \mid C \in \{D, \forall\mathcal{B}(q).D\}, \forall\mathcal{B}(q).D \in \mathsf{fcl}(C_0,\mathcal{R}), \\
& \qquad \varepsilon \in L(\mathcal{B}(q)), s \in D^{\mathcal{I},\mathcal{Z}}, \mathcal{Z} \in \Phi, \\
& \qquad \Gamma = \mathsf{Var}(\mathsf{tv}(D)\cap\mathbf{N}_s)\} \\
& \cup \{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \mid C \in \mathsf{fcl}(C_0,\mathcal{R}) \text{ is of the form } C_1 \sqcap C_2 \\
& \qquad \text{or } C_1 \sqcup C_2 \text{ or } \{v\} \text{ with } v \in \mathsf{N}_V, \\
& \qquad \mathcal{Z} \in \Phi, s \in C^{\mathcal{I},\mathcal{Z}}, \Gamma = \mathsf{Var}(\mathsf{tv}(C)\cap\mathbf{N}_s)\} \\
& \cup \{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \mid C \in \mathsf{fcl}(C_0,\mathcal{R}) \text{ is of the form } \exists R.D \\
& \qquad \text{or } (\geqslant nR.D) \text{ or } (\leqslant nR.D), \\
& \qquad \mathcal{Z} \in \Phi, s \in C^{\mathcal{I},\mathcal{Z}}, \Gamma = \mathsf{Var}(\mathsf{tv}(D)\cap\mathbf{N}_s)\}
\end{aligned}
$$

Figure 7: Defining $\mathcal{L}'(s)$.

Now, we need to show that all (Pi) are satisfied. (P1), (P2), (P3), (P4), (P5), (P7), (P9), (P10), (P11), (P12) and (P13) follow from the semantics and definition of T and the fact that $\mathcal{I}$ is a model of C_0 and $\mathcal{R}$. For (P6), if a nominal schema $v \in \mathcal{L}(s)$, then $s \in v^{\mathcal{I},\mathcal{Z}}$. So by the semantics, $s \in (\mathsf{gr}_{\mathcal{Z}}(v))^{\mathcal{I},\mathcal{Z}}$ for every $\mathcal{Z} \in \Phi$. Hence, for every $o \in \mathbf{N}_o$, $s \in o^{\mathcal{I}}$ and thus, $o \in \mathcal{L}(s)$. For (P8), if for some $v \in \mathbf{N}_s$, $v \in \mathcal{L}(s)$ and $\mathsf{gr}_{\mathcal{Z}}(\neg v) \in \mathcal{L}(s)$ for some $\mathcal{Z} \in \Phi$, then we would have both $s \in o^{\mathcal{I},\mathcal{Z}}$ and $s \in (\neg o)^{\mathcal{I},\mathcal{Z}}$ which is impossible. So (P8) must hold.

(P14) holds since whenever $\forall R.C \in \mathcal{L}(s)$, $s \in (\forall R.C)^{\mathcal{I},\mathcal{Z}}$ for all $\mathcal{Z} \in \Phi$, and hence $\forall\mathcal{B}_R.C \in \mathcal{L}(s)$ by definition of $\mathcal{L}(s)$.

For (P15), suppose $\forall\mathcal{B}(p).C \in \mathcal{L}(s)$, $\langle s,t\rangle \in \mathcal{E}(S) = S^{\mathcal{I}}$ for some $t \in \mathbf{S}$ and there exists $p \xrightarrow{S} q \in \mathcal{B}(p)$. We need to show that $\forall\mathcal{B}(q) \in \mathcal{L}(t)$. Assume the opposite (for contradiction), i.e., $\forall\mathcal{B}(q) \notin \mathcal{L}(t)$. By definition of $\mathcal{L}$, we have that either (i) $t \notin (\forall S_2.\cdots.\forall S_n.C)^{\mathcal{I},\mathcal{Z}}$ for some $S_2\cdots S_n \in L(\mathcal{B}(q))$ and some $\mathcal{Z} \in \Phi$; or (ii) $\varepsilon \in L(\mathcal{B}(q))$ and $t \notin C^{\mathcal{I},\mathcal{Z}}$ for some $\mathcal{Z} \in \Phi$. For case (i), $S_2\cdots S_n \in L(\mathcal{B}(q))$ implies that $SS_2\cdots S_n \in L(\mathcal{B}(p))$. Definition of $\mathcal{L}(s)$ implies that $s \in (\forall S.\forall S_2.\cdots.\forall S_n.C)^{\mathcal{I},\mathcal{Z}}$. Since $\langle s,t\rangle \in S^{\mathcal{I}}$, it must be the case that $t \in (\forall S_2.\cdots.\forall S_n.C)^{\mathcal{I},\mathcal{Z}}$ which is a contradiction. For (ii), it follows that $S \in L(\mathcal{B}(p))$. Hence, $s \in (\forall S.C)^{\mathcal{I},\mathcal{Z}}$. This implies that $t \in C^{\mathcal{I},\mathcal{Z}}$ which is also a contradiction.

For (P16), suppose $\forall\mathcal{B}(p).C \in \mathcal{L}(s)$ and $\varepsilon \in L(\mathcal{B}(p))$. By definition of $\mathcal{L}(s)$, $s \in C^{\mathcal{I},\mathcal{Z}}$, which also implies that $\{\mathsf{gr}_{\mathcal{Z},\Gamma}(\forall\mathcal{B}(p).C, \mathsf{gr}_{\mathcal{Z}_\Gamma}(C)\} \subseteq \mathcal{L}(s)$ for every $\mathcal{Z} \in \Phi$ where $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s)$ again by definition of $\mathcal{L}(s)$.

For (P17), suppose $\exists R.C \in \mathcal{L}(s)$. Let $\mathcal{Z} \in \Phi$ be any v-assignment. Definition of $\mathcal{L}'(s)$ means that $s \in (\exists R.C)^{\mathcal{I},\mathcal{Z}}$ which also implies that $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\exists R.C) \in \mathcal{L}(s)$. Hence, there exists $t \in \mathbf{S}$ with $\langle s,t\rangle \in R^{\mathcal{I}}$ and $t \in C^{\mathcal{I},\mathcal{Z}}$. So by definition of the mapping $\mathcal{L}$, $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \in \mathcal{L}(t)$ with $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s)$ as desired. (P18), (P19), and (P20) are analogous. □

We now have formalized the notion of tableau that corresponds to a model of a concept w.r.t. an RBox. This will be used to establish the soundness and completeness of the algorithm. Intuitively speaking, what we will do is establish a correspondence between tableaux and complete, clash-free completion graphs.

The following obvious lemma is useful for establishing the soundness.

Lemma 13. *Let $\Gamma \subseteq \mathsf{N}_V$ be a set of variables and Φ be the set of all full v-assignments. Let* Γ-coincidence *be a binary relation defined on Φ as follows where $\mathcal{Z}, \mathcal{Z}' \in \Phi$: $\mathcal{Z}$* Γ-coincides *with $\mathcal{Z}'$ iff $\forall x \in \Gamma : \mathcal{Z}(x) = \mathcal{Z}'(x)$. Then Γ-coincidence is an equivalence relation.*

Due to this lemma, we can partition Φ into disjoint sets of v-assignment such that v-assignments in one partition can be represented by its Γ-restriction. Whenever $\mathcal{Z}$ Γ-coincides with $\mathcal{Z}'$, $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) = \mathsf{gr}_{\mathcal{Z}'_\Gamma}(C)$ for any concept C. So, each choice of $\mathcal{Z}_\Gamma$ actually covers every possible v-assignment in the same partition induced by Γ.

For the following proof, a node y is a *direct heir* of a node x if x was merged into y. Additionally, the relation of being an *heir* is the transitive closure of the direct hair relation.

Lemma 14 (Soundness)**.** *If the expansion rules are applied to a $\mathcal{SROIQV}$ concept C_0 and a reduced RBox $\mathcal{R}$ such that they result in a complete and clash-free completion graph, then C_0 is satisfiable w.r.t. $\mathcal{R}$.*

Proof. It suffices to show that if the expansion rules can be applied to C_0 and $\mathcal{R}$ such that they yield a complete and clash-free completion graph, then there exists a tableau for C_0 and $\mathcal{R}$.

Let $\mathbf{G} = (V, E, \mathcal{L}_{\mathbf{G}}, \neq)$ be a complete and clash-free completion graph obtained by applying the expansion rules to C_0 and $\mathcal{R}$. We use standard unraveling of $\mathbf{G}$ to construct the appropriate tableau. For a node x that is directly blocked, denote $b(x)$ as the node y that blocks x.

A *path* p is a sequence of pairs of blockable nodes, called *path element*, of the form $[\langle x_1, x'_1\rangle, \ldots, \langle x_n, x'_n\rangle]$ where all x_i, x'_i are blockable nodes. For such a p, we define $\tau(p) := x_n$ and $\tau'(p) := x'_n$. Furthermore, for such a p, the path $[p|\langle x, x'\rangle]$ is the path obtained by augmenting p with $\langle x, x'\rangle$ at the end of the sequence, i.e., the path $[\langle x_1, x'_1\rangle, \ldots, \langle x_n, x'_n\rangle, \langle x, x'\rangle]$. Let $\Omega(\mathbf{G})$ be the set of all nominal nodes in $\mathbf{G}$. The set $\Pi(\mathbf{G})$ of all paths in $\mathbf{G}$ is inductively defined as:

- if x is a root node (a blockable node without an ancestor), or a successor of a nominal node, then $[\langle x, x\rangle] \in \Pi(\mathbf{G})$;
- if $p \in \Pi(\mathbf{G})$, a node y is blockable and not blocked, and $\langle \tau(p), y\rangle \in E$, then $[p|\langle y, y\rangle] \in \Pi(\mathbf{G})$;
- if $p \in \Pi(\mathbf{G})$, a node y is blocked, and $\langle \tau(p), y\rangle \in E$, then $[p|\langle b(y), y\rangle] \in \Pi(\mathbf{G})$.

Observe that for each path element $\langle x, y\rangle$: x is not blocked, y is blocked iff $x \neq y$, y is never indirectly blocked, and $\mathcal{L}_{\mathbf{G}}(x) = \mathcal{L}_{\mathbf{G}}(y)$.

We now construct a tableau $T = (\mathbf{S}, \mathcal{L}, \mathcal{E})$ for C_0 and $\mathcal{R}$ from $\mathbf{G}$. We use the following notations: $s \rightarrow_R t$ (resp. $s \Rightarrow_R t$) to denote that t is an R-successor (resp. R-neighbor) of s. We first set $\mathbf{S} := \Omega(\mathbf{G}) \cup \Pi(\mathbf{G})$. Next, we set:

$$\mathcal{L}(s) = \begin{cases} \mathcal{L}_G(s) & \text{if } s \in \Omega(\mathbf{G}) \\ \mathcal{L}_G(\tau(s)) & \text{if } s \in \Pi(\mathbf{G}) \end{cases}$$

$$\begin{aligned} \mathcal{E}(R) = \{&\langle s,t\rangle \mid s,t \in \Omega(\mathbf{G}), s \Rightarrow_R t\} \\ \cup \{&\langle s,p\rangle \mid s \in \Omega(\mathbf{G}), p \in \Pi(\mathbf{G}), s \Rightarrow_R \tau(p)\} \\ \cup \{&\langle p,s\rangle \mid s \in \Omega(\mathbf{G}), p \in \Pi(\mathbf{G}), \tau(p) \Rightarrow_R s\} \\ \cup \{&\langle p,q\rangle \mid p,q \in \Pi(\mathbf{G}), \text{ and} \\ &\quad \text{either } q = [p \mid \langle x,y\rangle] \text{ and } \tau(p) \rightarrow_R y \\ &\quad \text{or } p = [q \mid \langle x,y\rangle] \text{ and } \tau(q) \rightarrow_{\mathsf{Inv}(R)} y\} \end{aligned}$$

We now show that T is indeed a tableau for C_0 w.r.t. $\mathcal{R}$. Like earlier proofs, we use Φ to denote the finite set of all full v-assignments from N_V to N_I.

The algorithm implies that there is an heir x_0 of r_0 with $C_0 \in \mathcal{L}_{\mathbf{G}}(x_0)$. Note that s_0 cannot be blocked because either it is a nominal node or a blockable node that is a root of some tree of blockable nodes. Hence, it is never removed by the algorithm. So, there is some $s \in \mathbf{S}$ with $C_0 \in \mathcal{L}(s)$. We verify that T satisfies each (Pi).

(P1), and (P3) follow from definition of $\mathcal{E}$, R-neighbor and R-successor. (P2) is due to clash-freeness of $\mathbf{G}$ and definition of $\mathcal{E}$. (P4) is because each nominal is associated to a node in the initialization and furthermore, nominal nodes are never removed. (P5) is established because $\mathbf{G}$ is complete and clash-free — so o-rule is not applicable and no two nodes x and y with $x \neq$ share the same nominal label — and every nominal node is not unraveled into path as described above.

For (P6), since $\mathbf{G}$ is complete, GR1-rule is no longer applicable. Moreover, since Φ is finite, all possible choices of $\mathcal{Z}$ have been considered for an application of the GR1-rule. Hence, if a nominal schema v occurs in $\mathcal{L}(x)$, then all of its groundings, i.e., all of the nominals from $\mathbf{N}_o$ must also be in $\mathcal{L}(x)$. The rest follows from the definition of T.

(P7) follows from the clash-freeness of $\mathbf{G}$. Meanwhile, (P8) follows since $\mathbf{G}$ is complete and clash-free, hence all applications of GR1-rule never result in an occurrence of both o and $\neg o$ in the $\mathcal{L}(s)$ for some $o \in \mathbf{N}_o$.

(P9) is trivial since every time a node is constructed, $\top$ is always added to its labeling, and furthermore $\mathbf{G}$ is clash-free which means no node has $\bot$ in its labeling. (P10) is ensured by the completeness of $\mathbf{G}$ (sr-rule is no longer applicable). In addition, (P11) is ensured by the clash-freeness of $\mathbf{G}$.

(P12) follows since $\mathbf{G}$ is complete. More precisely, for $\Gamma = \mathsf{Var}(\mathsf{tv}(C \sqcap D) \cap \mathbf{N}_s)$, if $\Gamma = \emptyset$, then $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C \sqcap D) = C \sqcap D$, and thus, $\sqcap$-rule would added both conjuncts to $\mathcal{L}(s)$.

On the other hand, if $\Gamma \neq \emptyset$, the completeness of $\mathbf{G}$ implies that GR1-rule has been exhaustively applied and $\sqcap$-rule has been applied to all of the resulting partial groundings. Note that the soundness of the GR1-rule is guaranteed by Lemma 13 which implies that exhaustively considering every possible variation of Γ-restricted v-assignments suffices to ensure all full v-assignments are accounted for. (P13) is analogous.

(P14) is due to the completeness of $\mathbf{G}$, while (P16) is analogous to (P12). For (P15), suppose $\langle s, t\rangle \in \mathcal{E}(R)$, $\forall\mathcal{B}(p).C \in \mathcal{L}_{\mathbf{G}}(s)$ and $p \xrightarrow{R} q \in \mathcal{B}(p)$. If at least one of $s, t \in \Omega(\mathbf{G})$, then $\forall\mathcal{B}(q).C \in \mathcal{L}(t)$ follows from the definition of T and the completeness of $\mathbf{G}$. Otherwise, assume that $s, t \in \Pi(\mathbf{G})$. Then $\forall\mathcal{B}(p).C \in \mathcal{L}(\tau(s))$ by definition of T. Moreover, either $\tau(s) \rightarrow_R \tau'(t)$ or $\tau(t) \rightarrow_{\mathsf{Inv}(R)} \tau'(s)$. In the first case, the completeness of $\mathbf{G}$ ensures that $\forall(q).C \in \mathcal{L}_{\mathbf{G}}(\tau'(t))$. The definition of path implies that either $\tau'(t) = \tau(t)$ or $\tau(t)$ blocks $\tau'(t)$ which by the blocking condition, means that $\mathcal{L}_{\mathbf{G}}(\tau(t)) = \mathcal{L}_{\mathbf{G}}(\tau'(t))$. $\forall(q).C \in \mathcal{L}(t)$ is then established because $\mathcal{L}(t) = \mathcal{L}_{\mathbf{G}}(\tau(t))$. The second case is analogous.

For (P17), suppose $\exists R.D \in \mathcal{L}(s)$ and let $\mathcal{Z} \in \Phi$ be any full v-assignment and $\Gamma = \mathsf{Var}(\mathsf{tv}(D) \cap \mathbf{N}_s)$. Assume now that $\Gamma \neq \emptyset$. The other case where $\Gamma = \emptyset$ follows exactly the same reasoning. Note that $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\exists R.D) = \exists R.\mathsf{gr}_{\mathcal{Z}_\Gamma}(D)$. Let $C := \mathsf{gr}_{\mathcal{Z}_\Gamma}(D)$, hence $\exists R.C = \exists R.\mathsf{gr}_{\mathcal{Z}_\Gamma}(D)$. Obviously, that $\mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s) = \emptyset$, hence, the completeness of $\mathbf{G}$ implies that $\exists$-rule would have been applied on $\exists R.C$ on s or another node that blocks it, in addition the application of the GR2-rule to ground $\exists R.D$ that yields $\exists R.C$ in the first place. We verify each case.

If $s \in \Omega(\mathbf{G})$, then both $\exists R.D, \exists R.C \in \mathcal{L}_{\mathbf{G}}(s)$ and the completeness of $\mathbf{G}$ implies that there is an R-successor t with $C \in \mathcal{L}_{\mathbf{G}}(t)$. Here, if $t \in \Omega(\mathbf{G})$, then by definition of T, $\langle s, t\rangle \in \mathcal{E}(R)$ and $C \in \mathcal{L}(t)$. Otherwise, $t \in \Pi(\mathbf{G})$ and thus, t is a safe R-neighbor of s and not blocked. By definition of $\mathcal{E}$, there is a $p \in \Pi(\mathbf{G})$ for which $\tau(p) = t$, $\langle s, p\rangle \in \mathcal{E}(R)$ and $C \in \mathcal{L}(p)$.

Next, if $s \in \Pi(\mathbf{G})$, then $\{\exists R.D, \exists R.C\} \subseteq \mathcal{L}_{\mathbf{G}}(\tau(s))$, $\tau(s)$ is not blocked and since $\mathbf{G}$ is complete, there is an R-neighbor t of $\tau(s)$ with $C \in \mathcal{L}_{\mathbf{G}}(t)$. In this case, if $t \in \Omega(\mathbf{G})$, then $t \in \mathbf{S}$, $\langle s, t\rangle \in \mathcal{E}(R)$ and $C \in \mathcal{L}(t)$. On the other hand, if t is blockable and a successor of $\tau(s)$, then $[s|\langle \hat{t}, t\rangle] \in \mathbf{S}$ where $\hat{t} = t$ or $\hat{t} = b(t)$, $\langle s, [s|\langle t', t\rangle]\rangle \in \mathcal{E}(R)$, and $C \in \mathcal{L}([s|\langle t', t\rangle])$. Finally, if t is blockable and a predecessor of $\tau(s)$, then we have that $s = [p|\langle t, t\rangle|\langle \tau(s), \tau'(s)\rangle]$, $C \in \mathcal{L}([p|\langle t, t\rangle])$, and $\langle s, [p|\langle t, t\rangle]\rangle \in \mathcal{E}(R)$. In all cases, (P17) is satisfied for a given $\mathcal{Z} \in \Phi$. Finally, since $\mathbf{G}$ is complete, all full v-assignments have been considered for the application of the GR2-rule as ensured by Lemma 13. So, the above indeed holds for every $\mathcal{Z} \in \Phi$.

For (P18), assume that $(\geqslant nR.D) \in \mathcal{L}(s)$. Let $\mathcal{Z} \in \Phi$ be any full v-assignment and $\Gamma = \mathsf{Var}(\mathsf{tv}(D) \sqcap \mathbf{N}_s)$. Let $C := \mathsf{gr}_{\mathcal{Z}_\Gamma}(D)$, hence $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\geqslant nR.D) = (\geqslant nR.C)$. Obviously, $\mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s) = \emptyset$. Assume that $\Gamma \neq \emptyset$. The case where $\Gamma = \emptyset$ is analogous. First, consider the case where $s \in \Omega(\mathbf{G})$. Hence, $\{(\geqslant nR.D), (\geqslant nR.C)\} \subseteq \mathcal{L}_{\mathbf{G}}(s)$ by the completeness of $\mathbf{G}$ and definition of T. Completeness of $\mathbf{G}$ implies that there are n safe R-neighbors $t_1, \ldots, t_n$ of s with $t_i \mathrel{\dot{\neq}} t_j$ for $i \neq j$ and $C \in \mathcal{L}_{\mathbf{G}}(t_i)$. Definition of T ensures that each t_i corresponds to a $t'_i \in \mathbf{S}$ with $t'_i \neq t'_j$ for all $i \neq j$. Now, if t_i is blockable, since it is a safe R-neighbor of s, it is not blocked. So, a path $[p|\langle t_i, t_i\rangle] \in \mathbf{S}$ exists and $\langle s, [p|\langle t_i, t_i\rangle]\rangle \in \mathcal{E}(R)$. If t_i is a nominal node, then $\langle s, t_i\rangle \in \mathcal{E}(R)$.

Next is the case where $s \in \Pi(\mathbf{G})$. Hence, $\{(\geqslant nR.D), (\geqslant nR.C)\} \subseteq \mathcal{L}_{\mathbf{G}}(\tau(s))$. By the completeness of $\mathbf{G}$, there are n R-neighbors $t_1, \ldots, t_n$ of $\tau(s)$ such that $t_i \mathrel{\dot{\neq}} t_j$ for each $i \neq j$ and $C \in \mathcal{L}_{\mathbf{G}}(t_i)$ for $1 \leqslant i \leqslant n$. Like above, each of these t_i's correspond to some unique element of $\mathbf{S}$. If t_i is not safe, then $\langle s, t_i\rangle \in \mathcal{E}(R)$. Otherwise, t_i can be blocked whenever it is a successor of $\tau(s)$. Hence the path construction ensures that although $b(t_i) = b(t_j)$, there are two different elements of $\mathbf{S}$: $[p|\langle b(t_i), t_i\rangle] \neq [p|\langle b(t_j), t_j\rangle]$.

So, for (P18), all t_i's are different in all cases. Construction of T then implies that $C \in \mathcal{L}(t_i)$ for $1 \leqslant i \leqslant n$. Furthermore, the above holds for every possible full v-assignment $\mathcal{Z} \in \Phi$ which is ensured by the completeness of $\mathbf{G}$ and Lemma 13.

For (P19), assume that $(\leqslant nR.D) \in \mathcal{L}(s)$. Let $\mathcal{Z} \in \Phi$ be any full v-assignment and $\Gamma = \mathsf{Var}(\mathsf{tv}(D) \sqcap \mathbf{N}_s)$. Let $C := \mathsf{gr}_{\mathcal{Z}_\Gamma}(D)$, hence $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\leqslant nR.D) = (\leqslant nR.C)$. Clearly, $\mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s) = \emptyset$. Assume that $\Gamma \neq \emptyset$. The case where $\Gamma = \emptyset$ is analogous. By completeness, $\{(\leqslant nR.D), (\leqslant nR.C)\} \subseteq \mathcal{L}(s)$. Since $\mathbf{G}$ is clash-free, there are at most n R-neighbors t_i of s with $C \in \mathcal{L}(t_i)$. By construction of T, each $y \in \mathbf{S}$ such that $\langle s, y\rangle \in \mathcal{E}(R)$ corresponds to one (and not more) R-neighbor t_i of s or $\tau(s)$. Since $\mathcal{L}(y) = \mathcal{L}(t_i)$, we establish (P19).

(P20) is satisfied because $\mathbf{G}$ is complete, and whenever $\langle s, t\rangle \in \mathcal{E}(R)$, then t corresponds to either an R-neighbor of s, when $s \in \Omega(\mathbf{G})$, or of $\tau(s)$, when $s \in \Pi(\mathbf{G})$. $\square$

Lemma 15 (Completeness)**.** *If a $\mathcal{SROIQV}$ concept C_0 is satisfiable w.r.t. a reduced RBox $\mathcal{R}$, then the expansion rules can be applied to the initial completion graph of $(C_0, \mathcal{R})$ such that they yield a complete and clash free completion graph of $(C_0, \mathcal{R})$.*

Proof. Here, we show that for a given tableau $T = (\mathbf{S}, \mathcal{L}, \mathcal{E})$ for C_0 w.r.t. $\mathcal{R}$, we can apply the expansion rules from the initial completion graph for C_0 and $\mathcal{R}$ in such a way that they yield a complete and clash-free completion graph $\mathbf{G} = (V_{\mathbf{G}}, E_{\mathbf{G}}, \mathcal{L}_{\mathbf{G}}, \dot{\neq}_{\mathbf{G}})$.

We define a mapping $\mu\colon V_{\mathbf{G}} \rightarrow \mathbf{S}$ for $\mathbf{G}$ satisfying the following conditions (referred to as †), by induction on the rule applications:

- $\mathcal{L}_{\mathbf{G}}(x) \cap (\mathsf{fcl}(C_0, \mathcal{R}) \cup \mathsf{gcl}(C_0, \mathcal{R})) \subseteq \mathcal{L}(\mu(x))$ for each node x;
- for each pair of nodes x, y and role R, if y is an R-neighbor of x, then $\langle \mu(x), \mu(y)\rangle \in \mathcal{E}(R)$; and
- if $x \mathrel{\dot{\neq}_{\mathbf{G}}} y$, then $\mu(x) \neq \mu(y)$.

Let $T = (\mathbf{S}, \mathcal{L}, \mathcal{E})$ be a tableau for C_0 w.r.t. $\mathcal{R}$. Then, $C_0 \in \mathcal{L}(s)$ for some $s_0 \in \mathbf{S}$ and by (P4), for each nominal $o_1, \ldots, o_n$, there is a node $s_i \in \mathbf{S}$ such that $o_i \in \mathcal{L}(s_i)$, $1 \leqslant i \leqslant n$. Hence, the algorithm construct an initial completion graph $\mathbf{G}_0 = (V_0, E_0, \mathcal{L}_0, \dot{\neq}_0)$ with $V_0 = \{r_0, r_1, \ldots, r_n\}$, $E_0 = \emptyset$, $\dot{\neq}_0 = \emptyset$, $\mathcal{L}_0(r_0) = \{C_0, \top\}$, and $\mathcal{L}_0(r_i) = \{o_i, \top\}$, $1 \leqslant i \leqslant n$. So, we initially define $\mu_0(r_i) := s_i$ for $0 \leqslant i \leqslant n$. Conditions † are clearly satisfied.

Now, for inductive construction of μ, let $\mathbf{G} = (V_{\mathbf{G}}, E_{\mathbf{G}}, \mathcal{L}_{\mathbf{G}}, \dot{\neq}_{\mathbf{G}})$ be a completion graph and μ be a mapping satisfying †. We verify after each rule application that μ still satisfies † or can be extended to satisfy †.

$\sqcap$-rule: Let $C_1 \sqcap C_2 \in \mathcal{L}_{\mathbf{G}}(x)$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C_1 \sqcap C_2) \cap \mathbf{N}_s) = \emptyset$. Then $C_1 \sqcap C_2 \in \mathcal{L}(\mu(x))$, and by (P12), it holds for every v-assignment $\mathcal{Z}$ that, $\{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C_1), \mathsf{gr}_{\mathcal{Z}_\Gamma}(C_2)\} = \{C_1, C_2\} \subseteq \mathcal{L}(\mu(x))$. So, applying the $\sqcap$-rule does not violate †.

$\sqcup$-rule: Let $C_1 \sqcup C_2 \in \mathcal{L}_{\mathbf{G}}(x)$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C_1 \sqcap C_2) \cap \mathbf{N}_s) = \emptyset$. Then $C_1 \sqcup C_2 \in \mathcal{L}(\mu(x))$, and by (P13), we have for every v-assignment $\mathcal{Z}$, that $\{\mathsf{gr}_{\mathcal{Z}_\Gamma}(C_1), \mathsf{gr}_{\mathcal{Z}_\Gamma}(C_2)\} \cap \mathcal{L}(\mu(x)) = \emptyset$. Since in this case, $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C_1) = C_1$ and $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C_2) = C_2$, we obtain $\{C_1, C_2\} \cap \mathcal{L}(\mu(x))$. So, the $\sqcup$- rule can be applied without violating †.

sr-rule: Suppose $\exists R.\mathsf{Self} \in \mathcal{L}_{\mathbf{G}}(x)$. Then, $\exists R.\mathsf{Self} \in \mathcal{L}(\mu(x))$. By (P10), $\langle \mu(x), \mu(x) \rangle \in \mathcal{E}(R)$. Applying the sr-rule makes x become an R-neighbor of itself, hence † are not violated.

$\exists$-rule: Suppose $\exists R.C \in \mathcal{L}_{\mathbf{G}}(x)$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s) = \emptyset$. Then $\exists R.C \in \mathcal{L}(\mu(x))$. Also, by (P17) and (P9), for every v-assignment $\mathcal{Z}$, there is some $s \in \mathbf{S}$ such that $\langle \mu(x), s \rangle \in \mathcal{E}(R)$ and $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \in \mathcal{L}(s)$. But, $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) = C$ since $\Gamma = \emptyset$. Application of the $\exists$-rule generates a new node y, adds an edge $\langle x, y \rangle$, sets $\mathcal{L}_{\mathbf{G}}(\langle x, y \rangle) := \{R\}$, and sets $\mathcal{L}_{\mathbf{G}}(y) := \{C, \top\}$. So, μ can be extended without violating † by setting $\mu(y) := s$.

$\forall_1$-rule: Suppose $\forall R.C \in \mathcal{L}_{\mathbf{G}}(x)$. Then $\forall R.C \in \mathcal{L}(\mu(x))$, and by (P14), $\forall \mathcal{B}_R.C \in \mathcal{L}(\mu(x))$. Applying the $\forall_1$-rule clearly does not violate †.

$\forall_2$-rule: Suppose $\forall \mathcal{B}(p).C \in \mathcal{L}_{\mathbf{G}}(x)$, and $p \xrightarrow{R} q \in \mathcal{B}(p)$. If x has an R-neighbor y, then $\langle \mu(x), \mu(y) \rangle \in \mathcal{E}(R)$ by †. By (P15), $\forall \mathcal{B}(q).C \in \mu(y)$. So applying the $\forall_2$-rule will not violate †.

$\forall_3$-rule: Assume $\forall \mathcal{B}.C \in \mathcal{L}_{\mathbf{G}}(x)$, $\varepsilon \in L(\mathcal{B})$ and $\Gamma = \mathsf{tv}(C) \cap \mathbf{N}_s = \emptyset$. Then, $\forall \mathcal{B}.C \in \mu(x))$. By (P16), for every v-assignment $\mathcal{Z}$, $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \in \mathcal{L}(\mu(x))$. Since $\Gamma = \emptyset$, we must have that $C \in \mathcal{L}(\mu(x))$. Hence, the $\forall_3$-rule can be applied without violating †.

$\geqslant$-rule: Suppose $(\geqslant nR.C) \in \mathcal{L}_{\mathbf{G}}(x)$, and $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s) = \emptyset$. Then, $(\geqslant nR.C) \in \mathcal{L}(\mu(x))$. Due to (P18), for every v-assignment $\mathcal{Z}$, there are at least n different elements $t_i \in \mathbf{S}$, $1 \leqslant i \leqslant n$, such that $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) \in \mathcal{L}(t_i)$ and $\langle \mu(x), t_i \rangle \in \mathcal{E}(R')$ for some $R' \in L(\mathcal{B}_R)$. By Lemma 9.2, $R' \sqsubseteq^{*} R$. So, we have for $1 \leqslant i \leqslant n$ that $\langle \mu(x), t_i \rangle \in \mathcal{E}(R)$ due to (P3), $C \in \mathcal{L}(t_i)$ because $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) = C$ due to $\Gamma = \emptyset$, and $\top \in \mathcal{L}(t_i)$ because of (P9). Application of the $\geqslant$-rule would introduce n new nodes y_i for $1 \leqslant i \leqslant n$, add edges $\langle x, y_i \rangle$, set $\mathcal{L}_{\mathbf{G}}(x, y_i) := \{R\}$ and $\mathcal{L}_{\mathbf{G}}(y_i) := \{C, \top\}$, and assert $y_j \dot{\neq} y_k$ for $1 \leqslant j < k \leqslant n$. Thus, we extend μ without violating † by setting $\mu(y_i) := t_i$ for $1 \leqslant i \leqslant n$.

ch-rule: Suppose $(\leqslant nR.C) \in \mathcal{L}_{\mathbf{G}}(x)$, $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s) = \emptyset$, and x has some R-neighbor y. Then, $(\leqslant nR.C) \in \mathcal{L}(\mu(x))$ and $\langle \mu(x), \mu(y) \rangle \in \mathcal{E}(R)$. (P20) and $\Gamma = \emptyset$ imply that for every v-assignment $\mathcal{Z}$, either $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) = C \in \mathcal{L}(\mu(y))$ or $\mathsf{gr}_{\mathcal{Z}_\Gamma}(\mathsf{NNF}(\neg C)) = \mathsf{NNF}(\neg C) \in \mathcal{L}(\mu(y))$ but not both. If the ch-rule is applicable, then before the application, neither C nor $\mathsf{NNF}(\neg C)$ is in $\mathcal{L}_{\mathbf{G}}(y)$, and after the application, either $C \in \mathcal{L}_{\mathbf{G}}(y)$ or $\mathsf{NNF}(\neg C) \in \mathcal{L}_{\mathbf{G}}(y)$. Hence, † are not violated.

$\leqslant$-rule: Suppose $(\leqslant nR.C) \in \mathcal{L}_{\mathbf{G}}(x)$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s) = \emptyset$. Then, $(\leqslant nR.C) \in \mathcal{L}(\mu(x))$. According to (P19) and the fact that $\Gamma = \emptyset$, for every v-assignment $\mathcal{Z}$, there are at most n elements $t_i \in \mathbf{S}$, $1 \leqslant i \leqslant n$ for which there is an $R_i \in L(\mathcal{B}_R)$ with $\langle \mu(x), t_i \rangle \in \mathcal{E}(R_i)$ and $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) = C \in \mathcal{L}(t_i)$. Note by Lemma 9.2 and (P3), it follows that each t_i above satisfies $\langle \mu(x), t_i \rangle \in \mathcal{E}(R)$. If the $\leqslant$-rule is applicable, there are at least $n+1$ R-neighbor y_i, $1 \leqslant i \leqslant n+1$, such that $C \in \mathcal{L}(y_i)$. Thus, we have that $\langle \mu(x), \mu(y_i) \rangle \in \mathcal{E}(R)$ for each y_i. So, $\mu(y_1), \ldots, \mu(y_{n+1}) \in \{t_1, \ldots, t_n\}$ and hence, there must be two nodes y_j and y_k for which $\mu(y_j) = \mu(y_k)$, and this implies that $y_j \dot{\neq} y_k$ cannot hold. Furthermore, applying $\leqslant$-rule on any such two nodes y_j, y_k ensures that the resulting merged node retains all the concepts which were existing in either $\mathcal{L}_{\mathbf{G}}(y_j)$ or $\mathcal{L}_{\mathbf{G}}(y_k)$. Hence, the $\leqslant$-rule can be applied without violating †.

o-rule: If for some $o \in \mathbf{N}_o$, $o \in \mathcal{L}_{\mathbf{G}}(x) \cap \mathcal{L}_{\mathbf{G}}(y)$, then $o \in \mathcal{L}(\mu(x)) \cap \mathcal{L}(\mu(y))$. But (P5) ensures that actually $\mu(x) = \mu(y)$ which implies that $x \dot{\neq} y$ does not hold. So, applying the o-rule does not violate † because merging x and y retains all concepts which were existing in either $\mathcal{L}_{\mathbf{G}}(x)$ or $\mathcal{L}_{\mathbf{G}}(y)$.

NN-rule: Suppose $\{(\leqslant nR.C), o\} \subseteq \mathcal{L}_{\mathbf{G}}(x)$ for some $o \in \mathbf{N}_o \cup \mathbf{N}_a$, $\Gamma = \mathsf{Var}(\mathsf{tv}(C) \cap \mathbf{N}_s) = \emptyset$. Note that before the NN-rule is applied to x, $(\leqslant nR.C) \in \mathsf{fcl}(C_0, \mathcal{R}) \cup \mathsf{gcl}(C_0, \mathcal{R})$ since additional at-most restrictions not in $\mathsf{fcl}(C_0, \mathcal{R}) \cup \mathsf{gcl}(C_0, \mathcal{R})$ are only introduced by the NN-rule, and furthermore, the NN-rule is applicable only once on x for a particular concept $(\leqslant nR.C)$. Hence, $(\leqslant nR.C) \in \mathcal{L}(\mu(x))$. By (P19) and the fact that R is simple (thus using Lemma 9.2 and (P3), for every v-assignment $\mathcal{Z}$, there are m different elements $t_i \in \mathbf{S}$ with $m \leqslant n$, $1 \leqslant i \leqslant m$, and $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C) = C \in \mathcal{L}(t_i)$ (due to $\Gamma = \emptyset$). Thus, if the NN-rule is applicable, it can be applied in such a way that, guessing $k := m$, $(\leqslant mR.C)$ is added to $\mathcal{L}_{\mathbf{G}}(x)$, m new nominal nodes $y_1, \ldots, y_m$ are created such that $\{C, o_i, \top\} \subseteq \mathcal{L}_{\mathbf{G}}(y_i)$, $1 \leqslant i \leqslant m$, and $y_i \dot{\neq} y_j$ are asserted for all $1 \leqslant i < j \leqslant m$. The mapping μ can be safely extended by setting $\mu(y_i) := t_i$ for $1 \leqslant i \leqslant m$.

G1-rule: Suppose $D = \{v\} \in \mathbf{N}_s$ or $D = C_1 \sqcap C_2$ or $D = C_1 \sqcup C_2$, $D \in \mathcal{L}_{\mathbf{G}}(x)$ and $\Gamma = \mathsf{Var}(\mathsf{tv}(D) \cap \mathbf{N}_s)$ with $\Gamma \neq \emptyset$. Then, $D \in \mathsf{fcl}(C_0, \mathcal{R})$ which implies that $D \in \mathcal{L}(\mu(x))$. Due to (P6), $o \in \mathcal{L}(\mu(x))$ for every $o \in \mathbf{N}_o$. Hence, the GR1-rule that adds a particular nominal o' to $\mathcal{L}_{\mathbf{G}}(x)$ can be applied without violating †. Similarly, with $D = C_1 \sqcap C_2$, the GR1-rule would add $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C_1 \sqcap C_2)$ to $\mathcal{L}_{\mathbf{G}}(x)$. This does not violate † due to (P12). Finally, if the GR1-rule adds $\mathsf{gr}_{\mathcal{Z}_\Gamma}(C_1 \sqcup C_2)$, then no violation of † happens due to (P13).

G2-rule and G3-rule: Similar to the G1-rule.

Completeness can now be obtained as follows. Given a tableau T for C_0 w.r.t. $\mathcal{R}$, an initial completion graph $\mathbf{G}$

can be created as described above. Moreover, the mapping μ can be initialized based on the initial completion graph and μ is initially well-defined since it is guaranteed by the tableau properties as described in the initial construction of μ. Next, the above construction ensures that whenever a rule is applied to $\mathbf{G}$, † always holds. Any sequence of rule applications terminates, hence we obtain that $\mathbf{G}$ is complete. In addition, $\mathbf{G}$ is clash-free due to the following reasons.

First, $\bot \notin \mathcal{L}_{\mathbf{G}}(x)$ for every node x, as otherwise (P9) will be violated and the tableau would not exist. Secondly, $\mathbf{G}$ cannot contain a node x such that $\{C, \neg C\} \subseteq \mathcal{L}_{\mathbf{G}}(x)$ because that would mean $\{C, \neg C\} \subseteq \mathcal{L}(\mu(x))$ by †, and this would violate (P7). Thirdly, application of the sr-rule never leads to a clash of the form (3) due to (P11). Next, a clash of the form (4) cannot happen because of (P2). A clash of the form (5) cannot occur because otherwise, it would contradict (P19). Finally, a clash of the form (6) also cannot occur due to (P5).

Hence, together with Lemma 12, if C_0 is satisfiable w.r.t. a reduced RBox $\mathcal{R}$, then the expansion rules can be applied such that it yields a complete and clash-free completion graph. □

Conclusion

We have presented a tableau algorithm for the description logic $\mathcal{SROIQV}$. It improves on the only known algorithm to date, which is based on full grounding, by applying grounding in a selective and delayed fashion. We have provided correctness results and an example which shows the advantages of our approach over full grounding, by significantly pruning the size of the tableau which needs to be constructed. While the basic idea of our approach seems to be intuitively simple (just ground when needed), details of the algorithm are elaborate and required non-trivial modifications of previous work on tableau algorithms for description logics.

In terms of realization of reasoning with nominal schemas, this paper is only the beginning of our investigations. While our algorithm provides a flexible way of delayed and selective grounding, for implementations it will be necessary to obtain good heuristics for grounding choices. We also have the feeling that the technical notion of concept top level (Definintion 10), which featured heavily in our proofs and restricts flexibility in delaying grounding, can possibly be weakened significantly. Finally, other automated reasoning approaches, e.g., such based on resolution, may provide even better algorithmizations for $\mathcal{SROIQV}$ or prominent fragments thereof.

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