

# Formal Concept Analysis and Resolution in Algebraic Domains

Pascal Hitzler and Matthias Wendt

Artificial Intelligence Institute, Dresden University of Technology

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# Overview

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R&Z 2001

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## Coherent algebraic cpos

*cpo*: directed complete partial order with bottom  $(D, \sqsubseteq)$

$c \in \mathbf{K}(D)$  (compact) iff  $(\forall A \text{ directed})(d \sqsubseteq \bigsqcup A \implies (\exists a \in A)d \sqsubseteq a)$

*cpo algebraic*:  $(\forall x)(x = \bigsqcup(x \downarrow \cap \mathbf{K}(D)))$

*Scott topology*: base  $\{\uparrow c \mid c \in \mathbf{K}(D)\}$

*coherent*: finite intersections of compact-opens are compact-open

Examples: Finite posets with bottom. Powersets.  $\mathbb{T}^\omega$ .

## Smyth powerdomain as ideal completion

$X, Y \subseteq \mathcal{K}(D)$ .  $X \sqsubseteq^\# Y$  iff  $(\forall y \in Y)(\exists x \in X)(x \sqsubseteq y)$

Smyth powerdomain of a coherent algebraic cpo: proper ideal completion of the set of all finite subsets of  $D$ , ordered by  $\sqsubseteq^\#$ .

Used for modelling nondeterminism in domain theory.

In the following:  $(D, \sqsubseteq)$  coherent algebraic domain.

## Logic RZ

(Rounds & Zhang, 2001)

*clause*  $X$ : finite subset of  $\mathcal{K}(D)$

$w \in D$ :  $w \models X$  iff  $(\exists x \in X)(x \sqsubseteq w)$ .

*theory*  $T$ : set of clauses.

$w \models T$  iff  $(\forall X \in T)(w \models X)$ .

$T \models X$  iff  $(\forall w \in D)(w \models T \implies w \models X)$ .

## Logic RZ

Proof theory: (WLP'02)

$$\overline{\{\perp\}}$$

$$\frac{X; \quad a \in X; \quad y \sqsubseteq a}{\{y\} \cup (X \setminus \{a\})}$$

$$\frac{X; \quad y \in K(D)}{\{y\} \cup X}$$

$$\frac{X_1 \quad X_2; \quad a_1 \in X_1 \quad a_2 \in X_2}{\text{mub}\{a_1, a_2\} \cup (X_1 \setminus \{a_1\}) \cup (X_2 \setminus \{a_2\})}$$

Logic RZ is compact.

## Smyth powerdomain via the logic RZ

(Rounds & Zhang 2001)

The logically closed theories are the ideals under  $\sqsubseteq^\#$ .

Smyth powerdomain: consistent closed theories under set-inclusion.

## Relation to Formal Concept Analysis (FCA)

$G$  set of objects;  $M$  set of attributes.  $C \subseteq G \times M$  formal context.

$A \subseteq G$  then  $A' = \{m \in M \mid (\forall g \in A)(g, m) \in C\}$ .

$B \subseteq M$  then  $B' = \{g \in G \mid (\forall m \in B)(g, m) \in C\}$ .

*Formal concept:* Pair  $(A, B)$  with  $A' = B$ ,  $A = B'$ .

Equivalently: All  $(B', B'')$  for  $B \subseteq M$ .

*Formal concept lattice:*

complete lattice of all concepts ordered by  $\supseteq$  in second argument.

## Relation to Formal Concept Analysis (FCA)

Consider subposet  $D$  of all object and attribute concepts.

*Galois subhierarchy*

Reversely ordered (add  $\perp$ ).

If  $D$  is a coherent algebraic cpo (eg. all finite cases), then

for  $K(D) \supseteq \{b_i \mid i \in I\} = B \subseteq M$

(i.e.  $B$  is a set of compact attribute concepts)

we have

$$B'' = \{b \in M \mid \{\{b_i\} \mid i \in I\} \models \{b\}\}.$$

i.e. the corresponding concept intent consists of the attributes derivable in the logic RZ as singleton clauses

## Logic programming in coherent algebraic domains

(Rounds & Zhang 2001)

Add material implication:  $X \leftarrow Y$  for  $X, Y$  clauses.

$w \models P$ : if  $w \models Y$  for  $X \leftarrow Y \in P$ , then  $w \models X$ .

Propagation rule  $\text{CP}(P)$ :

$$\frac{X_1 \quad \dots \quad X_n; \quad a_i \in X_i; \quad Y \leftarrow Z \in P; \quad \text{mub}\{a_1, \dots, a_n\} \models Z}{Y \cup \bigcup_{i=1}^n (X_i \setminus \{a_i\})}$$

Semantic operator on theories:

$$\mathcal{T}_P(T) = \text{cons}(\{Y \mid Y \text{ is a } \text{CP}(P)\text{-consequence of } T\}).$$

- ▶  $\mathcal{T}_P$  is Scott continuous [RZ01].
- ▶  $\text{fix}(\mathcal{T}_P) = \text{cons}(P)$ .

## Additon of default negation

Extended rules:  $X \leftarrow Y, \sim Z$ .

$P$  program,  $T$  theory. Define  $P/T$ :

Replace  $Y, \sim Z$  by  $Y$  if  $T \not\models Z$ .

Remove rule if  $T \models Z$ .

$T$  answer theory for  $P$  if  $T = \text{cons}(P/T) = \text{fix}(\mathcal{T}_{P/T})$ .

## A version of default logic

Consider  $\mathbb{T}^\omega$ .

Clauses are the propositional formulae in disjunctive normal form.

Extended rules correspond to defaults.

Answer theories correspond to default extensions.

But logical consequence is not classical.

► On  $\mathbb{T}^\omega$  we obtain something akin to propositional default logic.

## Answer set programming

We do the same with *models*.

$P$  program,  $w \in D$ . Define  $P/w$ :

Replace  $Y, \sim Z$  by  $Y$  if  $w \not\models Z$ .

Remove rule if  $w \models Z$ .

$w$  *min-answer model* for  $P$  if  $w$  is minimal with  $w \models \text{fix}(\mathcal{T}_{P/w})$ .

## Answer set programming

Consider  $\mathbb{T}^\omega$ .

Consider programs  $P$  with rules  $X \leftarrow Y, \sim Z$  such that:

$Y$  singleton clause

$X, Y, Z$  contain only atoms in  $\mathbb{T}^\omega$  or  $\perp$

These programs are exactly extended disjunctive programs.

Min-answer models  $w$  correspond to *answer sets*  $\{L \text{ atom} \mid w \models \{L\}\}$ .

## Further Work?