Multivalued Mappings, Fixed-Point Theorems and Disjunctive Databases

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Slides for presentation of the paper with the same title. The full paper will be published in the conference proceedings.

Disjunctive Logic Programs and Databases

A disjunctive logic program Π is

a finite set of clauses of the form

$$\forall(\underbrace{A_1 \vee \cdots \vee A_n}_{\texttt{head}} \leftarrow \underbrace{L_1 \wedge \cdots \wedge L_m}_{\texttt{body}})$$

* A_i atoms
* L_i literals

Written:

$$A_1, \ldots, A_n \leftarrow L_1, \ldots, L_m$$

clauses as above	*disjunctive* logic program
no negation	*definite* disjunctive log. prog.
$n = 1$ in every clause	*normal* logic program
no function symbols	*disjunctive database*

* B_Π set of all ground atoms in Π
* $I_\Pi = 2^{B_\Pi}$ set of all interpretations for Π

Identify Π with ground(Π).

Models via Fixed Points

▶ Define *single step operator* T_Π.

$T_\Pi : I_\Pi \to \mathcal{P}(I_\Pi)$ defined by $J \in T_\Pi(I)$ iff

- For each `head` $\leftarrow$ `body` in ground(Π) with $I \models$ `body` exists A in `head` with $A \in J$.
- For all $A \in J$ exists `head` $\leftarrow$ `body` in ground(Π) with $I \models$ `body` and A in `head`.

$*$ P non-disjunctive then T_P single-valued

Theorem

$I \in T_\Pi(I)$ (i.e. I is a *fixed point* of T_Π) iff

- I is a model for Π
- For every $A \in I$ exists `head` $\leftarrow$ `body` in ground(Π) s.t. A in `head` and $I \models$ `body`.

$*$ Call I with $I \in T_\Pi(I)$ a *supported model* of Π.

The Stable Model Semantics

(Gelfond & Lifschitz 1988, 1991)

▶ A model is stable if it can reproduce itself as follows.

Π definite disjunctive

> $I \in I_\Pi$ is an *answer set* iff
> for every `head` $\leftarrow$ `body` in ground(Π)
>
> - If $I \models$ `body` then exists A in `head` with $A \in I$.
>   (I is *closed by rules in* Π)
> - I is minimal closed by rules in Π

$*$ $\alpha(\Pi)$ set of all answer sets of Π

Π normal disjunctive, $I \in I_\Pi$

> - Apply program transformation Π^I:
>   For every clause `head` $\leftarrow$ `body` in ground(Π)
>   - If $\neg A$ in `body` for $A \in I$ then remove clause.
>   - Otherwise remove all negative literals
>   and keep remaining clause.
> - Operator $GL(I) = \alpha(\Pi^I)$
> - Every $I \in GL(I)$ is a *stable model* for Π.

Example for Stable Model Semantics

Program Π:

$$p(0) \vee q(0) \leftarrow$$
$$p(a) \vee q(0) \leftarrow q(0) \wedge \neg p(0)$$

▶ Case $p(0) \notin I$.

Program Π^I:

$$p(0) \vee q(0) \leftarrow$$
$$p(a) \vee q(0) \leftarrow q(0)$$

$* \ \alpha(\Pi^I) = \{ \quad \{p(0)\}, \quad \{q(0)\} \quad \}$
$* \ \{q(0)\} \in \alpha(\Pi^{\{q(0)\}})$

▶ Case $p(0) \in I$.

Program Π^I:

$$p(0) \vee q(0) \leftarrow$$

$* \ \alpha(\Pi^I) = \{ \quad \{p(0)\}, \quad \{q(0)\} \quad \}$
$* \ \{p(0)\} \in \alpha(\Pi^{\{p(0)\}})$

$\Rightarrow \{p(0)\}, \{q(0)\}$ are the stable models of Π

Normal Derivatives I

▶ Derive normal program from disjunctive program.

Relates
* normal and disjunctive programs
* their semantics
* their single-step operators
* single- and multi-valued operators

Given disjunctive program Π.

> A *normal derivative* of Π
> is a (ground) normal program P satisfying
>
> - For every `head` $\leftarrow$ `body` in ground(Π)
>   exists $A \leftarrow$ `body` in P
>   with A in `head`.
> - For every $A \leftarrow$ `body` in P
>   exists `head` $\leftarrow$ `body` in ground(Π)
>   with A in `head`.

* I.e. all clauses in P are derived using the first rule.
* Π' set of all normal derivatives of Π

Normal Derivatives II

Theorem:

$J \in T_\Pi(I)$ if and only if
$J = T_P(I)$ for some $P \in \Pi'$.

Corollaries:

- If any normal derivative of Π has a supported model then Π has a supported model.
- If Π has a supported model then some normal derivative has a supported model.
- If any normal derivative of Π is any of acceptable, locally stratified, definite, etc. then Π has a supported model.

Proposition:

Π ground disj. database consisting of n rules with d_k conjunctions ($k = 1, \ldots, n$), respectively.
Set $N = \prod_{k=1}^{n}(2^{d_k} - 1)$.

- $|\Pi'| \leq N$
- For any $I \in I_\Pi$ we have $|T_\Pi(I)| \leq N$.

$*$ The bound N is sharp in both cases.

Normal Derivatives and Stable Models I

Theorem:

If I is a stable model for Π
then there is $P \in \Pi'$ such that $T_P(I) = I$.

Corollary:

Any stable model is a supported model.

Proposition:

If $P \in \Pi'$ and $I = T_P(I)$
then I is closed by rules in Π^I.

Corollary:

I is a stable model for Π if and only if
it is minimal with the property
that it is a supported model of some $P \in \Pi'$.

Normal Derivatives and Stable Models II

Proof of Theorem:

We construct $P \in \Pi'$ following the construction of Π^I.
Let $\texttt{head} \leftarrow \texttt{body}$ in ground(Π).
N set of all atoms occurring as negative literals in $\texttt{body}$.
R set of all atoms occurring as positive literals in $\texttt{body}$.

I. $N \cap I \neq \emptyset$
Choose atom A in $\texttt{head}$
and add clause $A \leftarrow \texttt{body}$ to P.
Note: $I \not\models \texttt{body}$.

II. $N \cap I = \emptyset$
Then $\texttt{head} \leftarrow R$ is in Π^I
and $R \subseteq I \Longrightarrow \texttt{head} \cap I \neq \emptyset$ holds.

II.1. $R \not\subseteq I$
Choose atom A in $\texttt{head}$
and add clause $A \leftarrow \texttt{body}$ to P.
Note: $I \not\models \texttt{body}$.

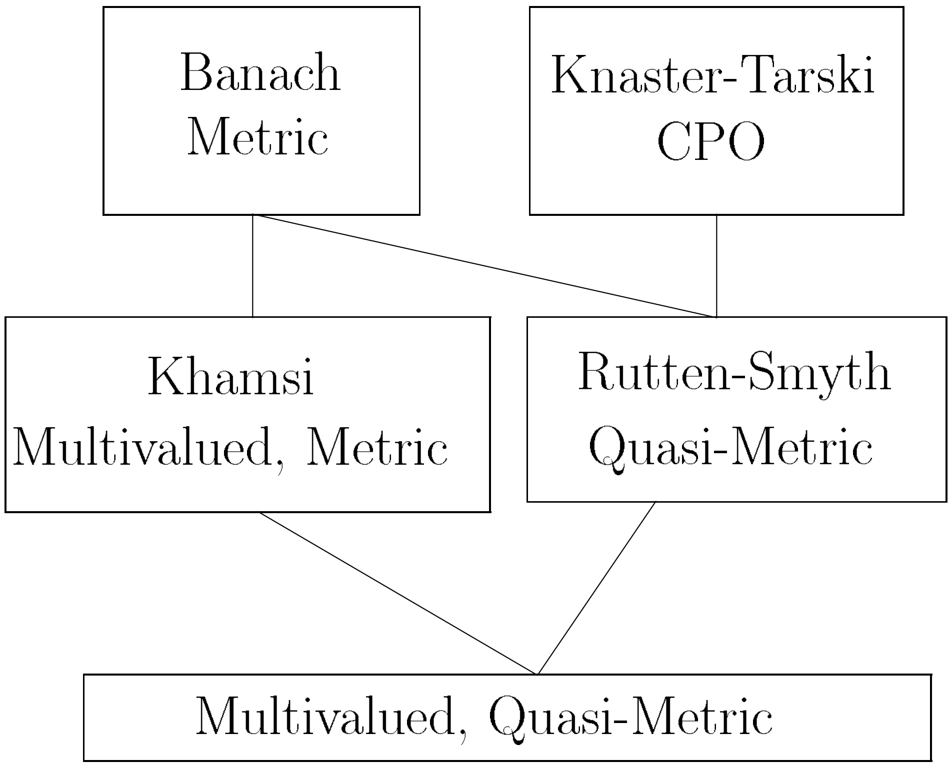
II.2. $R \subseteq I$
For each A in $\texttt{head}$ with $A \in I$
add clause $A \leftarrow \texttt{body}$ to P.
Note: $I \models \texttt{body}$.

Normal Derivatives and Stable Models III

Proof of Theorem continued:
We obtain: $P \in \Pi'$ and $T_P(I) \subseteq I$.
Remains to show: $T_P(I) = I$.
Assume $T_P(I) \subset I$, i.e. $A \in I \setminus T_P(I)$.
For each `head` $\leftarrow$ `body` in ground(Π) with $R \subseteq I$
we have A not in `head`.
But then $J = I \setminus \{A\}$ is closed by rules in Π^I.
This contradicts minimality of I. ■

A General Multi-Valued Fixed-Point Theorem I

Dependencies between some fixed-point theorems.



(X, d), $d : X \times X \to \mathbb{R}$ is a *quasi-metric space* if for all $x, y, z \in X$

- $d(x, x) = 0$
- $d(x, y) = 0 = d(y, x)$ implies $x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$

A General Multi-Valued Fixed-Point Theorem II

(x_n) is a *Cauchy-sequence* (*CS*) if
for all $\varepsilon > 0$ exists $n_0 \in \mathbb{N}$ s.t. for all $n \geq m \geq n_0$
$d(x_m, x_n) < \varepsilon$.
A CS (x_n) *converges* to $x \in X$ if
for all $y \in X$, $d(x, y) = \lim d(x_n, y)$.
(X, d) is *CS-complete* if every CS converges.
$f : X \to \mathcal{P}(X)$ is

- *non-expanding* if
  for all $x, y \in X$, $a \in T(x)$
  exists $b \in T(y)$ with $d(a, b) \leq d(x, y)$.
- *contracting* if
  exists $0 \leq k < 1$ such that
  for all $x \neq y \in X$, $a \in T(x)$
  exists $b \in T(y)$ with $d(a, b) \leq kd(x, y)$.
- *orbitally CS-continuous* if
  for every (x_n) with $x_{n+1} \in f(x_n)$ for all n
  which is a Cauchy-sequence
  we have $\lim x_n \in f(\lim x_n)$.

A General Multi-Valued Fixed-Point Theorem III

Theorem:

Each of the following conditions
ensures the existence of a fixed point of f
if (X, d) is a CS-complete quasi-metric space.

- $f : X \to \mathcal{P}(X)$ is
  - a contraction and
  - orbitally CS-continuous.
- $f : X \to \mathcal{P}(X)$ is
  - non-empty,
  - non-expanding,
  - exists $x \in X$ and $y \in f(x)$ with $d(x, y) = 0$, and
  - orbitally CS-continuous.