The Well-Founded Semantics Is a Stratified Fitting Semantics

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Contents

Uniform treatment of different semantics in logic programming/nonmonotonic reasoning.

In particular: Fitting semantics made more credulous using "stratification" yields well-founded semantics.

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Notation

We work on ground instantiations of normal logic programs.

Negation symbols may appear in clause bodies.

Essentially, program P is countably infinite set of propositional rules.
Herbrand base $B_P \approx$ set of propositional variables (atoms).

$$A \leftarrow B_1, \ldots, B_n, \neg C_1, \ldots, \neg C_m$$

$$A \leftarrow \texttt{body}$$

(We talk about Prolog only in a very abstract sense.)

Level mappings

Level mapping $B_P \to \alpha$ for some ordinal α.

- order on atoms
- precedence
- dependence
- distance

How level mappings were used before

- Termination under resolution.
  Bezem, Apt & Pedreschi, Marchiori, Ruggieri

- Classifying programs with respect to semantic properties.
  Cavedon, Apt & Blair & Walker, Przymusinski, van Gelder

- Topological (metric) analysis.
  Fitting, Hitzler & Seda

- Relation to connectionist systems.
  Hölldobler & Kalinke & Störr, Hitzler & Seda

Now: Uniform approaches to different semantics.

Least models

Positive (definite) program P.

There is a unique model M of P for which there exists a level mapping $l : B_P \to \alpha$ such that for each $A \in B_P$ with $M \models A$ there exists $A \leftarrow \texttt{body}$ in P with $M \models \texttt{body}$ and $l(A) > l(B)$ for each $B \in \texttt{body}$.

$M = T_P{\uparrow}\omega = \mathrm{lfp}(T_P)$ is the least model of P.

$l(A) = \min\{n \mid A \in T_P{\uparrow}(n+1)\}$.

Stable models

(Fages 1994)

P normal (with negation).

A model M of P is stable if and only if there exists a level mapping $l : B_P \rightarrow \alpha$ such that for each $A \in B_P$ with $M \models A$ there exists $A \leftarrow \texttt{body}$ in P with $M \models \texttt{body}$ and $l(A) > l(B)$ for all $B \in \texttt{body}^+$.

$\texttt{body}^+$: all atoms occuring positively in $\texttt{body}$.

$M = \mathrm{GL}_P(M) = T_{P/M}\uparrow\omega = \mathrm{lfp}(T_{P/M})$.

$l(A) = \min\{n \mid A \in T_{P/M}\uparrow(n+1)\}$.

Kleene's strong three-valued logic

Truth values $f < u < t$, $\wedge = \min$, $\vee = \max$, $\neg$ as expected.

Interpretations: consistent signed sets of atoms.

$I = I^+ \dot{\cup} \neg I^- \subseteq B_P \cup \neg B_P$.

I^+: true atoms

I^-: false atoms

Signed: contains atoms and negated atoms.

Consistent: does not contain both A and $\neg A$.

With order $I \subseteq K$: Plotkin's domain $\mathbb{T}^\omega$.

I-partial level mapping:

partial mapping $l : B_P \to \alpha$ with $\mathrm{dom}(l) = I^+ \cup I^-$.

Set $l(\neg A) = l(A)$.

Fitting models

There is a greatest model M of P such that there is an M-partial level mapping l for P such that each $A \in \mathrm{dom}(l)$ satisfies one of the following conditions.

(Fi) $A \in M$ and there exists $A \leftarrow L_1, \dots, L_n$ in P such that for all i we have $L_i \in M$ and $l(A) > l(L_i)$.

(Fii) $\neg A \in M$ and for each $A \leftarrow L_1, \dots, L_n$ in P there exists i with $\neg L_i \in M$ and $l(A) > l(L_i)$.

$M = \Phi_P{\uparrow}\alpha = \mathrm{lfp}(\Phi_P)$ Fitting model.

$l(A) = \min\{\beta \mid A \in \Phi_P{\uparrow}(\beta + 1)\}$.

Fitting semantics stratified

Replace

(Fii) $\neg A \in M$ and for $A \leftarrow L_1, \ldots, L_n$ in P
there exists i with $\neg L_i \in M$ and $l(A) > l(L_i)$.

by

(WFii) $\neg A \in M$ and for each $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in P
one of the following holds:

(WFiia) There exists i with $\neg A_i \in M$ and $l(A) \geq l(A_i)$.

(WFiib) There exists j with $B_j \in M$ and $l(A) > l(B_j)$.

Prevent recursion through negation: Idea behind (local) stratification.
(Apt, Blair & Walker; Przymusinski)

Well-founded models

There is a greatest model M of P such that there is an M-partial level mapping l for P such that each $A \in \mathrm{dom}(l)$ satisfies one of the following conditions.

(Fi) $A \in M$ and there exists $A \leftarrow L_1, \ldots, L_n$ in P such that for all i we have $L_i \in M$ and $l(A) > l(L_i)$.

(WFii) $\neg A \in M$ and for each $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in P one of the following holds:

(WFiia) There exists i with $\neg A_i \in M$ and $l(A) \geq l(A_i)$.

(WFiib) There exists j with $B_j \in M$ and $l(A) > l(B_j)$.

$M = W_P \uparrow \alpha = \mathrm{lfp}(W_P)$ well-founded model.

$l(A) = \min\{\beta \mid A \in W_P \uparrow (\beta + 1)\}$.

Well-founded models

Replacing (WFi) by

(SFi) $A \in M$ and there exists $A \leftarrow A_1 \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in P such that $A_i, \neg B_j \in M$, $l(A) \geq l(A_i)$, $l(A) > l(B_j)$.

is not satisfactory.

Greatest model may not exist.

Program $p \leftarrow p$, $q \leftarrow \neg p$
has two incomparable models $\{p, \neg q\}$, $\{\neg p, q\}$.

$l(p) = 0$, $l(q) = 1$.

Weakly perfect models (weak stratification)

There is a greatest model M of P such that there is an M-partial level mapping l for P such that each $A \in \mathrm{dom}(l)$ satisfies one of the following conditions.

(Fi) $A \in M$ and there exists $A \leftarrow L_1, \dots, L_n$ in P such that for all i we have $L_i \in M$ and $l(A) > l(L_i)$.

(WSii) $\neg A \in M$ and for each $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ in P one of the following holds:

- (WSiia) There exists i with $\neg A_i \in M$ and $l(A) > l(A_i)$.
- (WSiib) There exists i with $\neg A_i \in M$, for all k we have $l(A) \geq l(A_k)$, for all j we have $l(A) > l(B_j)$.
- (WSiic) There exists j with $B_j \in M$ and $l(A) > l(B_j)$.

Slight technical modification of *weak stratification*.

Nontrivial proof, 4 pages in length.

Another characterization of the well-founded model

Due to Lifschitz, McCain, Przymusinski and Stärk 1995.

The well-founded model is the least model M of P such that there exists a level mapping $l : B_P \to \mathbb{N}$ such that for each $A \notin M^-$ there exists a rule $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, B_m$ in P such that

(i) for all i: $l(A) > l(A_i)$ and $A \notin M^-$

(ii) for all j: $B_j \notin M$.

Conditions on the atoms which are not false.

Alternating fixed-point semantics

stable models: $M = \mathrm{GL}_P(M) = T_{P/M}{\uparrow}\omega$.

GL_P antitonic, GL_P^2 monotonic.

well-founded model:

$$\mathrm{lfp}\left(\mathrm{GL}_P^2\right) \cup \neg\mathrm{gfp}\left(\mathrm{GL}_P^2\right) \quad = \quad \mathrm{lfp}\left(\mathrm{GL}_P^2\right) \cup \neg\mathrm{GL}_P\left(\mathrm{lfp}\left(\mathrm{GL}_P^2\right)\right).$$

$L_\alpha = \mathrm{GL}_P^2{\uparrow}\alpha \qquad G_\alpha = \mathrm{GL}_P(L_\alpha)$.

$l(A) = (\alpha, n)$ with:

For $A \in \mathrm{lfp}\left(\mathrm{GL}_P^2\right)$: α least with $A \in L_{\alpha+1}$
n least with $A \in T_{P/G_\alpha}{\uparrow}(n+1)$.

For $A \notin \mathrm{gfp}\left(\mathrm{GL}_P^2\right)$: α least with $A \notin G_{\beta+1}$
$n = \omega$.

Quo Vadis?

Generalize

extended disjunctive etc. programs.

Apply

computation of models (answer set programming).