

# Dislocated Topologies

Pascal Hitzler and Anthony Seda

Slides for Presentation at SCAM 2000, Bratislava,  
Slovak Republic, April 2000

## Contents

- Motivation: Denotational Semantics
- A Generalized Banach Theorem
- Dislocated Topologies
- Dislocated Metrics
- Application: Acceptable Programs
- Conclusions

Slides for presentation of the paper with the same title.

The paper will be submitted to the proceedings.

## Motivation: Denotational Semantics

### Classical Approach

Given (part of) a computer program  $P$ .

Associate operator  $T_P$  with it, which is

- monotonic,
- continuous in the Scott topology and
- therefore has a least fixed point  $M_P$  by the Knaster-Tarski theorem.

$M_P$  is interpreted as the *denotational semantics* (i.e. the mathematical meaning) of  $P$ .

### Logic Programming with negation

Associated operators are often not monotonic.

Above mentioned approach is invalid.

### Possible solution:

Use alternative fixed-point theorems.

## A Generalized Banach Theorem I

$X$  set,  $\rho : X \times X \rightarrow \mathbb{R}_0^+$  function. Consider the following conditions for all  $x, y, z \in X$ .

(Mi)  $\rho(x, x) = 0$

(Mii) if  $\rho(x, y)(= \rho(y, x)) = 0$  then  $x = y$

(Miii)  $\rho(x, y) = \rho(y, x)$

(Miv)  $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

(Miv')  $\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\}$

$\rho$	(Mi)	(Mii)	(Miii)	(Miv)
metric	x	x	x	x
pseudo-metric	x		x	x
quasi-metric	x	x		x
<i>dislocated metric</i>		x	x	x

Dislocated metric abbreviated as *d-metric*;  
called *metric domains* by Matthews (1985).

Strong triangle inequality (Miv') holds:  
prefix *ultra* (e.g. ultrametric, d-ultrametric).

## A Generalized Banach Theorem II

Due to Matthews (1985).

We will give alternative proof later.

**Definitions** on d-metric spaces  $(X, \rho)$

*Convergence:*  $x_n \rightarrow x$  if  $\rho(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ .

— Limits are unique.

*Cauchy sequence: (CS)*

$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall m, n \geq n_0 : \rho(x_m, x_n) < \varepsilon$ .

— Converging sequences are Cauchy sequences.

$(X, \rho)$  *complete* if all CS converge.

$f : X \rightarrow X$  *contraction* if exists  $0 \leq \lambda < 1$

s.t.  $\rho(f(x), f(y)) \leq \lambda \rho(x, y)$  for all  $x, y$ .

### **Generalized Banach Theorem**

$(X, \rho)$  complete d-metric space.

$f : X \rightarrow X$  contraction.

Then  $f$  has a unique fixed point.

## Dislocated Topologies I

We investigate d-metrics  
from a topological point of view.

### Neighbourhoods

Let  $X$  be a set.

Let  $\vartriangleleft \subseteq X \times \mathcal{P}(X)$  be a *d-membership relation*  
i.e.  $x \vartriangleleft A \subseteq B$  implies  $x \vartriangleleft B$  for all  $A, B \subseteq X$ .

For all  $x \in X$  let  $\emptyset \neq \mathcal{U}_x \subseteq \mathcal{P}(X)$ .

$(X, \{\mathcal{U}_x \mid x \in X\}, \vartriangleleft)$  is a *d-topological space*  
if for all  $x \in X$  we have:

(Ni)  $U \in \mathcal{U}_x$  implies  $x \vartriangleleft U$ .

(Nii)  $U, V \in \mathcal{U}_x$  implies  $U \cap V \in \mathcal{U}_x$ .

(Niii)  $\forall U \in \mathcal{U}_x \exists V \subseteq U \forall y \vartriangleleft V : V \in \mathcal{U}_x \& U \in \mathcal{U}_y$ .

(Niv)  $U \in \mathcal{U}_x$  and  $U \subseteq V$  imply  $V \in \mathcal{U}_x$ .

Each  $U \in \mathcal{U}_x$  called a *d-neighbourhood* of  $x$ .

$(X, \varrho)$  d-metric space.

$B_\varepsilon(x) = \{y \in X \mid \varrho(x, y) < \varepsilon\}$  balls.

Define  $x \vartriangleleft A$  if exists  $\varepsilon > 0$  s.t.  $B_\varepsilon(x) \subseteq A$ .

Define  $\mathcal{U}_x = \{A \subseteq X \mid x \vartriangleleft A\}$ .

$(X, \{\mathcal{U}_x \mid x \in X\}, \vartriangleleft)$  is a d-topological space.

## Dislocated Topologies II

### Convergence

$(X, \mathcal{U}, \triangleleft)$  d-top. space,  $x \in X$ ,  $(x_\lambda)$  net.  
 $x_\lambda$  *d-converges* to  $x \in X$  if for all  $U \in \mathcal{U}_x$   
exists  $\lambda_0$  s.t.  $x_\lambda \in U$  for all  $\lambda > \lambda_0$ .

- ↪ Constant sequences may not converge.
- ↪ Convergence in a d-metric is exactly d-convergence in underlying d-topology.

### Continuity

$(X, \mathcal{U}, \triangleleft_X)$ ,  $(Y, \mathcal{V}, \triangleleft_Y)$  d-topological spaces.  
 $f : X \rightarrow Y$  is *d-continuous* at  $x_0 \in X$  if  
for each  $V \in \mathcal{V}_{f(x_0)}$  exists  $U \in \mathcal{U}_{x_0}$  s.t.  $f(U) \subseteq V$ .

$f$  *d-continuous* if d-continuous at  $x$  for all  $x \in X$ .

- ↪  $f : X \rightarrow Y$  d-continuous iff for each net  $x_\lambda$   
with limit  $x_0$ ,  $f(x_\lambda)$  is a net with limit  $f(x_0)$ .

## Dislocated Metrics I

### Continuity

- ↪  $X, Y$  d-metric spaces,  $f : X \rightarrow Y$  function.  
 $f$  d-continuous at  $x \in X$  iff for each  $\varepsilon > 0$   
exists  $\delta > 0$  s.t.  $f(B_\delta(x)) \subseteq B_\varepsilon(f(x_0))$ .
- ↪ Contractions are d-continuous.

**Proof** of Generalized Banach Theorem  
now analogous to proof in metric case.

### Obtaining d-Metrics from Metrics

$(X, \varrho)$  d-metric space.

$u_\varrho : X \rightarrow \mathbb{R} : x \mapsto \varrho(x, x)$  *dislocation function*.

- ↪  $u_\varrho$  is d-continuous.
- ↪  $(X, d)$  metric space,  $u : X \rightarrow \mathbb{R}_0^+$  function,  
 $T : \mathbb{R}^2 \rightarrow \mathbb{R}_0^+$  symmetric with triangle inequality.  
Then  $\varrho(x, y) := d(x, y) + T(u(x), u(y))$  d-metric.
- ↪ Furthermore,  
if  $u, T$  continuous and  $T(x, x) = x$  for all  $x$   
then if  $(X, d)$  complete, so is  $(X, \varrho)$ .

## Dislocated Metrics II

↪  $(X, d)$  ultrametric space,  $u : X \rightarrow \mathbb{R}_0^+$  function.

Then  $\varrho(x, y) := \max\{d(x, y), u(x), u(y)\}$  is d-ultrametric and  $\varrho(x, x) = u(x)$  for all  $x \in X$ .

↪ Furthermore, if  $u$  continuous then if  $(X, d)$  complete, so is  $(X, \varrho)$ .

**Examples** for d-metric spaces.

a)  $(\mathbb{R}_0^+, \varrho)$  with  $\varrho : (x, y) \rightarrow x + y$ .

Define  $T : \mathbb{R}_0^+ \times \mathbb{R}_0^+ : (x, y) \mapsto \frac{1}{2}(x + y)$ .

b)  $d(x, y) := \frac{1}{2}|x - y|$  metric on  $\mathbb{R}_0^+$ ,  $u = \text{id}_{\mathbb{R}_0^+}$ .

Then  $\varrho(x, y) := d(x, y) + T(u(x), u(y))$  d-metric with  $\varrho(x, y) = \max\{x, y\}$ .

c)  $\mathcal{I}$  set of all closed intervals on  $\mathbb{R}$ .

$d([a, b], [c, d]) := \frac{1}{2}(|a - c| + |b - d|)$  metric.

$u([a, b]) := b - a$ .

Then  $\varrho(I, J) := d(I, J) + T(u(I), u(J))$  d-metric with  $\varrho([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ .

## Application: Acceptable Programs I

### Logic Programs

Logic program: finite set of first-order clauses

$\forall(A \leftarrow L_1 \wedge \cdots \wedge L_n)$ , written  $A \leftarrow L_1, \dots, L_n$ .

Atom  $A$  *head*, literals  $L_1, \dots, L_n$  *body* of clause.

$B_P$  Herbrand base.

$I_P = \mathcal{P}(B_P)$  set of all interpretations.

*Single-step operator*  $T_P : I_P \rightarrow I_P$  defined as

$T_P(I)$  is set of all  $A \in B_P$  s.t. there exists ground instance  $A \leftarrow \text{body}$  of clause in  $P$  with  $I \models \text{body}$ .

$I$  is model of  $P$  iff  $T_P(I) \subseteq I$ .

$T_P$  not monotonic if negation occurs in  $P$ .

Fixed points of  $T_P$  are called *supported models*.

### Acceptable Programs (Apt & Pedreschi 1993)

$P$  logic program,  $p, q$  predicate symbols.

$p$  *refers to*  $q$  if there is clause in  $P$  with  $p$  in head,  $q$  in body.

$p$  *depends on*  $q$ : refl., trans. closure of “refers to”.

$\text{Neg}_P$ : pred. symbols from neg. literals in  $P$ .

$\text{Neg}_P^*$ : predicate symbols on which the predicate symbols in  $\text{Neg}_P$  depend.

## Application: Acceptable Programs II

$P^-$ : all clauses with head containing a predicate symbol from  $\text{Neg}_P^*$ .

$P$  called *acceptable* (wrt.  $I \in I_P, l : B_P \rightarrow \mathbb{N}$ ) if

- $I$  restricted to  $\text{Neg}_P^*$  is supported model of  $P^-$ ,
- for each ground instance  $A \leftarrow L_1, \dots, L_n$  of a clause in  $P$  and for all  $i$ ,

$I \models \bigwedge_{j=1}^{i-1} L_j$  implies  $l(A) > l(L_i)$ .

Define complete ultrametric on  $I_P$ :  $J, K \in I_P$ ,

$d(K, K) = 0$  and  $d(J, K) = 2^{-n}$ ,

where  $J, K$  differ on  $A$  with  $l(A) = n$

but coincide on all  $B$  with  $l(B) < n$ .

$f : I_P \rightarrow \mathbb{R} : f(K) = 0$  if  $K \subseteq I$  and

$f(K) = 2^{-n}$ , where  $n$  smallest such that exists

$A \in B_P$  with  $l(A) = n$ ,  $K \models A$  and  $I \not\models A$ .

$u : I_P \rightarrow \mathbb{R} : K \mapsto \max\{f(K'), d(K', I)\}$  where

$K'$  is  $K$  restricted to pred. symbols not in  $\text{Neg}_P^*$ .

•  $\varrho(J, K) := \max\{d(J, K), u(J), u(K)\}$

is complete d-ultrametric on  $I_P$ .

•  $T_P$  is contractive wrt.  $\varrho$ .

## Conclusions

- Results analogous to conventional topology.

d-uniformities?

d-open sets?

- Do these spaces appear elsewhere in Mathematics or Theoretical Computer Science?

Then further theoretical investigations needed.

- Apply to more general classes of logic programs.

- Interpret  $u_\rho(x)$  as *undesirability*.

Appropriate for settings other than Logic Programming?

- Even more general fixed-point theorems.

Merge with Priess-Crampe & Ribenboim (2000) Theorem?