A "Converse" of the Banach Contraction Mapping Theorem

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Metric spaces

A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying (Mi) to (Miv) for all $x, y, z \in X$. d is an *ultrametric* if it also satisfies (Mv).

(Mi) $d(x, y) = 0$ implies $x = y$.

(Mii) $d(x, x) = 0$.

(Miii) $d(x, y) = d(y, x)$.

(Miv) $d(x, z) \leq d(x, y) + d(y, z)$.

(Mv) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

A metric space is *complete* if every Cauchy sequence in it converges.

The Banach theorem

Let (X, d) be a complete metric space and let $f : X \to X$ be a function such that there exists $0 \leq \lambda < 1$ with $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.

Then $f^n(x)$ converges for any $x \in X$ to the unique fixed point a of f.

$f^0(x) = x$ for all $x \in X$.
$f^{n+1}(x) = f(f^n(x))$ for all $x \in X$.

The converse

Let (X, τ) be a T_1 topological space and $f : X \to X$ be a function which has a unique fixed point a and such that for each $x \in X$ we have that $f^n(x)$ converges to a in τ.

Then there exists a function $d : X \times X \to \mathbb{R}$ such that (X, d) is a complete ultrametric space and such that for all $x, y \in X$ we have $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$.

The proof, step 1

Choose $x \in X$. Define $T(x) \subseteq X$ inductively as follows:

(1) $x \in T(x)$.
(2) If $y \in T(x)$ with $y \neq a$, then $f(y) \in T(x)$.
(3) If $y \in T(x)$ then $z \in T(x)$ for all $z \in f^{-1}(y)$.

Now, let $\mathcal{T} = \{T(x) \mid x \in X\}$
and note that $T(x) = T(y)$ iff $T(x) \cap T(y) \neq \emptyset$.

The proof, step 2

For each $T \in \mathcal{T}$ define $l : T \to \mathbb{Z} \cup \{\infty\}$ inductively:

(1) If $a \in T$, then $l(a) = \infty$ and if $T \setminus \{a\}$ is nonempty, then choose an arbitrary $x \in T$ with $x \neq a$ and set $l(x) = 0$.

(2) For each $y \in T$ with $f(y) \neq a$ and $l(y) = k$, let $l(f(y)) = k + 1$.

(3) For each $y \in T$ with $y \neq a$ and $l(y) = k$ let $l(z) = k - 1$ for all $z \in f^{-1}(y)$.

l is well-defined since (X, τ) is a T_1 space.

The proof, step 3

Define a mapping $\iota : \mathbb{Z} \cup \{\infty\} \to \mathbb{R}$ by

$$\iota(k) = \begin{cases} 0 & \text{if } k = \infty, \\ 2^{-k} & \text{otherwise.} \end{cases}$$

Furthermore, define a mapping $\varrho : X \times X \to \mathbb{R}$ by

$$\varrho(x, y) = \max\{\iota(l(x)), \iota(l(y))\}$$

and a mapping $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} \varrho(x, y) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

The proof, step 4

(X, d) is an ultrametric space which is complete.

For $x \neq a$ we have $l(f(x)) = l(x) + 1$ by definition of l, and hence $\iota(l(f(x))) = \frac{1}{2}\iota(l(x))$.

For all $x, y \in X$ we have that $d(f(x), f(y)) \leq \frac{1}{2}d(x, y)$.

Remarks I

Any $x \neq a$ is an isolated point with respect to d i.e. $\{x\}$ is open and closed in the topology generated by d.

If (x_n) is a sequence in X which converges in d to some $x \neq a$, then the sequence (x_n) is eventually constant.

The metric d does not in general generate τ, but the iterates $(f^n(x))$ of f converge to a both with respect to τ and with respect to d.

ϱ is a dislocated ultrametric (cf. Hitzler & Seda, SCAM2000).

The converse is essentially a generalization of a proposition in (Hölldobler, Kalinke 1994), which was proven for finite X.

Remarks II

There exists a partial order $\leq$ on X such that the following hold:

(i) $(X, \leq)$ is chain complete, i.e. every increasing chain has a least upper bound.

(ii) f is monotonic i.e. if $x \leq y$ then $f(x) \leq f(y)$.

(iii) For each increasing chain (x_λ) in X we have $f(\sup_\lambda x_\lambda) = \sup_\lambda f(x_\lambda)$

(iv) $f(x) \geq x$ for all $x \in X$.

We define this partial order $\leq$ on X by (1) $x \leq a$ for all $x \in X$ and (2) $x \leq y$ if $l(x) \leq l(y)$ and $y \in T(x)$.

Note the similarity between these conditions and the hypothesis of the Kleene theorem.