

# BCU Mathematics Contest 1999

Edited by Finbarr Holland & Pascal Hitzler

March 13, 1999

# BCU MATHEMATICS CONTEST, MARCH 13, 1999

## SOLUTIONS TO THE TEAM CONTEST PAPER

1. Since a non-leap year has 365 days, we're required to find positive integers  $a, b$  such that

$$ab + a + b = 365.$$

Now  $ab + a + b = (a + 1)(b + 1) - 1$ , hence  $a, b$  are positive integral solutions of the equation

$$(a + 1)(b + 1) = 366 = 1.2.3.61.$$

Since  $a, b$  are positive, and the equation is symmetric in  $a, b$  the only possible solution pairs are  $(1, 182), (2, 121), (5, 60)$ . The answer is therefore 3.

Contestants who find these solutions by trial and error, and do not show that these are the only solutions, get a maximum of 4 points.

2. If

$$a_n = \frac{A}{n-1} + \frac{B}{n+1}, \quad n = 2, 3, \dots,$$

then, in particular,

$$a_2 = A + \frac{1}{3}B, \quad a_3 = \frac{1}{2}A + \frac{1}{4}B,$$

i.e.,

$$\frac{2}{3} = A + \frac{1}{3}B, \quad \frac{1}{4} = \frac{1}{2}A + \frac{1}{4}B,$$

whence

$$2 = 3A + B, \quad 1 = 2A + B.$$

Thus  $A = 1$  and  $B = -1$ . It remains to show that these values work for all  $n \geq 2$ . But this is easy: if  $n \geq 2$ , then

$$\begin{aligned} \frac{1}{n-1} + \frac{-1}{n+1} &= \frac{n+1 - (n-1)}{(n-1)(n+1)} \\ &= \frac{2}{n^2-1} \\ &= a_n. \end{aligned}$$

OR

$$\begin{aligned} \frac{2}{n^2-1} &= \frac{2}{(n-1)(n+1)} \\ &= \frac{A}{n-1} + \frac{B}{n+1} \\ &= \frac{A(n+1) + B(n-1)}{(n-1)(n+1)} \\ &= \frac{(A+B)n + (A-B)}{n^2-1}, \end{aligned}$$

for  $n \geq 2$  iff  $2 = (A+B)n + A - B$ ,  $n = 2, 3, \dots$ . This is so iff  $A = -B = 1$ .

We now have for  $k = 2, 3, \dots$

$$\begin{aligned} s_k &= \sum_{n=2}^k a_n \\ &= \sum_{n=2}^k \left[ \frac{1}{n-1} - \frac{1}{n+1} \right] \\ &= \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots \\ &\quad + \left( \frac{1}{k-2} - \frac{1}{k} \right) + \left( \frac{1}{k-1} - \frac{1}{k+1} \right). \end{aligned}$$

Removing brackets and cancelling we see that

$$s_k = 1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1} = \frac{3}{2} - \frac{2k+1}{k(k+1)}.$$

OR

$$\begin{aligned} s_k = \sum_{n=2}^k a_n &= \sum_{n=2}^k \left[ \frac{1}{n-1} - \frac{1}{n+1} \right] \\ &= \sum_{n=2}^k \left[ \left( \frac{1}{n-1} - \frac{1}{n} \right) + \left( \frac{1}{n} - \frac{1}{n+1} \right) \right] \\ &= \sum_{n=2}^k \left( \frac{1}{n-1} - \frac{1}{n} \right) + \sum_{n=2}^k \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{k} + \frac{1}{2} - \frac{1}{k+1} \\ &= \frac{3}{2} - \frac{2k+1}{k(k+1)}. \end{aligned}$$

OR

$$\begin{aligned} s_k = \sum_{n=2}^k a_n &= \sum_{n=2}^k \left[ \frac{1}{n-1} - \frac{1}{n+1} \right] \\ &= \sum_{n=2}^k \frac{1}{n-1} - \sum_{n=2}^k \frac{1}{n+1} \\ &= \sum_{n=1}^{k-1} \frac{1}{n} - \sum_{n=3}^{k+1} \frac{1}{n} \\ &= 1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1} \\ &= \frac{3}{2} - \frac{2k+1}{k(k+1)}. \end{aligned}$$

Clearly, then

$$s_{100} - s_{10} = \frac{21}{10.11} - \frac{201}{100.101} < \frac{21}{10.11} < \frac{1}{5}.$$

OR

$$s_{100} - s_{10} = \frac{1}{10} + \frac{1}{11} - \left( \frac{1}{100} + \frac{1}{101} \right) < \frac{2}{10} = \frac{1}{5},$$

and

$$s_{100} - s_{10} > \frac{2}{11} - \frac{2}{100} = \frac{2.89}{11.100} > \frac{4}{25}.$$

Thus the error lies between  $4/25$  and  $1/5$ .

3.

$$y' = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4) = 3(x - 2)^2 \geq 0, \forall x.$$

Hence  $y$  is increasing on  $(-\infty, \infty)$ . Also,

$$y'' = 6x - 12 = 6(x - 2).$$

Clearly,  $y'' \geq 0, \forall x \geq 2$  and  $y'' \leq 0, \forall x \leq 2$ . Hence  $y$  is convex if  $x \geq 2$  and concave if  $x \leq 2$ . It has no local extrema and one point of inflection at  $x = 2$ . Also,

$$\lim_{x \rightarrow \infty} y = \infty, \lim_{x \rightarrow -\infty} y = -\infty.$$

Since  $y$  is a cubic polynomial with real coefficients it crosses the  $x$ -axis at least once. To see where, note that

$$y = x^3 - 6x^2 + 12x - 7 = (x - 2)^3 + 1 = 0$$

if  $x = 1$ . (Alternatively, the root can be found by trial and error.)  
Then

$$y = (x - 1)(x^2 - 5x + 7) = (x - 1)\left((x - 5/2)^2 + \frac{3}{4}\right).$$

Hence  $y$  crosses the horizontal axis precisely once.

OR

It is  $x^3 - 6x^2 + 12x - 7 = x^3 - 3 \cdot 2x^2 + 3 \cdot 2^2x - 2^3 + 1 = (x - 2)^3 + 1$ .  
Therefore, the function has the equation

$$y = (x - 2)^3 + 1,$$

i.e. it is the cubic function  $y = x^3$  where the point of inflection is moved to  $(2, 1)$ . Hence, it is convex if  $x \geq 2$ , concave if  $x \leq 2$ , and has point of inflection  $(2, 1)$ .

4. Determine the area of the bounded region enclosed by the set of points in the  $xy$ -plane that satisfy the equation

$$xy + y^2 - yx^2 - x^3 = 0.$$

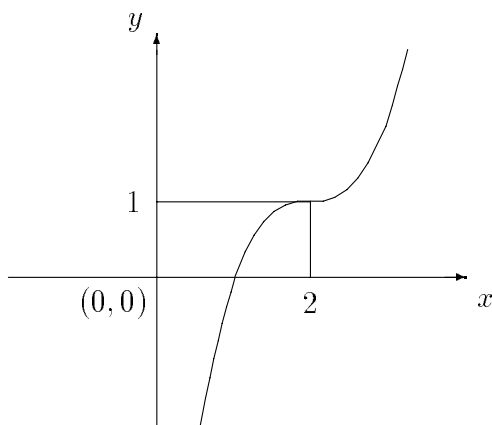


Figure 1: Graph of  $y = x^3 - 6x^2 + 12x - 7$

Note that  $xy + y^2 - yx^2 - x^3 = (y + x)(y - x^2)$ . Thus  $(x, y)$  belongs to the set

$$\{(x, y) : xy + y^2 - yx^2 - x^3 = 0\}$$

iff it lies on the line  $y = -x$  or on the parabola  $y = x^2$ . The *bounded* region enclosed by these curves is therefore the set

$$\{(x, y) : x^2 \leq y \leq -x, -1 \leq x \leq 0\},$$

the area of which is

$$\int_{-1}^0 (-x) dx - \int_{-1}^0 x^2 dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

OR

Not everybody will spot the factorization. Some may treat  $xy + y^2 - yx^2 - x^3 = 0$  as a quadratic in  $y$ — $y^2 + (x - x^2)y - x^3 = 0$ —and solve for  $y$ :

$$\begin{aligned} y &= \frac{-(x - x^2) \pm \sqrt{(x - x^2)^2 + 4x^3}}{2} \\ &= \frac{-(x - x^2) \pm \sqrt{x^2 + 2x^3 + x^4}}{2} \\ &= \frac{-(x - x^2) \pm |x + x^2|}{2}. \end{aligned}$$

Now denoting the roots by  $y_+, y_-$  we see that

$$y_+ = \begin{cases} x^2, & \text{if } x(1+x) \geq 0 \\ -x, & \text{if } x(1+x) \leq 0, \end{cases}$$

and

$$y_- = \begin{cases} -x, & \text{if } x(1+x) \geq 0 \\ x^2, & \text{if } x(1+x) \leq 0, \end{cases}$$

It follows that the set

$$\{(x, y) : xy + y^2 - yx^2 - x^3 = 0\}$$

is the union of the line  $y = -x$  and the parabola  $y = x^2$ . The problem can be finished as before.

Inevitably, some of those who treat it this way will say that  $\sqrt{(x+x^2)^2} = x+x^2$  and simplify the work for themselves! But they should be penalised!

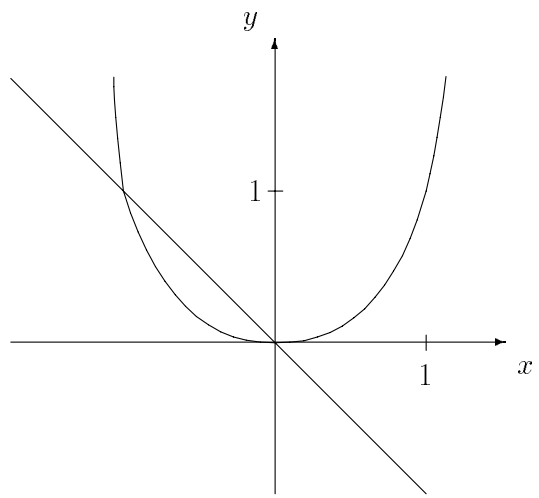


Figure 2: The two graphs enclosing the area in question.

# BCU MATHEMATICS CONTEST, MARCH 13, 1999

## SOLUTIONS TO THE INDIVIDUAL CONTEST PAPER

1. Picture:

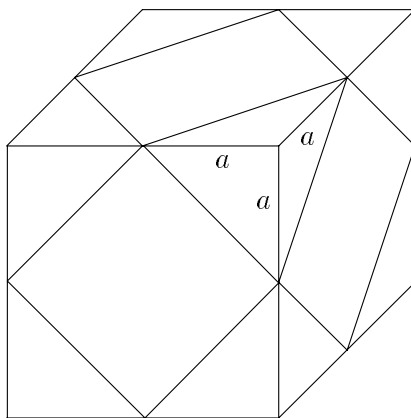


Figure 3: The cuts on the wooden cube.

Volume:

Any of the pyramids which is cut away at some edge has volume  $V_P = \frac{a^3}{6}$ . The volume of the remaining solid is therefore

$$V_R = 8a^3 - \frac{4}{3}a^3 = \frac{20}{3}a^3.$$

Surface:

Along the cuts appear equilateral triangles with sides of length  $a\sqrt{2}$ . Hence, each of those triangles has area  $A_T = \frac{a^2}{2}\sqrt{3}$ . The surface area

of the remaining solid is therefore

$$O_R = 24a^2 - 8 \cdot 3 \cdot \frac{a^2}{2} + 8 \cdot \frac{a^2}{2} \sqrt{3} = 12a^2 + 4a^2 \sqrt{3}.$$

2. The largest root of the equation is

$$\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

For  $a \rightarrow 0$ , both numerator and denominator tend to 0.

First solution:

$$\beta = \frac{(\sqrt{b^2 - 4ac} - b)(\sqrt{b^2 - 4ac} + b)}{2a(\sqrt{b^2 - 4ac} + b)} = \frac{-2c}{\sqrt{b^2 - 4ac} + b}$$

and therefore

$$\lim_{a \rightarrow 0} \beta = -\frac{c}{b}.$$

Second solution:

Using de l'Hospital's rule we obtain

$$\lim_{a \rightarrow 0} \beta = \lim_{a \rightarrow 0} \frac{\frac{-2c}{\sqrt{b^2 - 4ac}}}{2} = -\frac{c}{b}.$$

Third Solution:

Let  $h = -4ac$  and, for  $x > 0$ , define  $f(x) = \sqrt{x}$ . Then  $b = f(b^2)$  and

$$\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -2c \left( \frac{f(b^2 + h) - f(b^2)}{h} \right).$$

Now as  $a \rightarrow 0$ ,  $h \rightarrow 0$  and so

$$\begin{aligned} \lim_{a \rightarrow 0} \beta &= -2c \lim_{h \rightarrow 0} \frac{f(b^2 + h) - f(b^2)}{h} \\ &= -2c f'(b^2) \\ &= -2c \frac{1}{2\sqrt{b^2}} \\ &= -\frac{c}{b}, \end{aligned}$$

since

$$f'(x) = \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}, \quad \forall x > 0.$$

Comment. All three proofs rely on the crucial fact that  $f$  is continuous on  $[0, \infty)$ .

3. First Solution. Join  $B$  to  $E$  and to  $Y$ . Let  $\theta = \angle YAX$  and  $\phi = \angle ACX$ , so that  $\phi = \pi/2 - \theta$ . Then since  $\angle BXA = \angle BYA = \pi/2$  we see that  $\angle CBY = \theta$ . But the triangles  $BCY$  and  $BYE$  are similar. Hence  $\angle YBE = \angle CBY = \theta$  and  $\angle BEC = \phi$ . It now follows that

$$\angle ABE = \angle ABC + \angle CBY + \angle YBE = \phi - 2\theta + \theta + \theta = \phi.$$

Hence the triangle  $AEB$  is isosceles and so  $|AE| = |AB| = d$ .

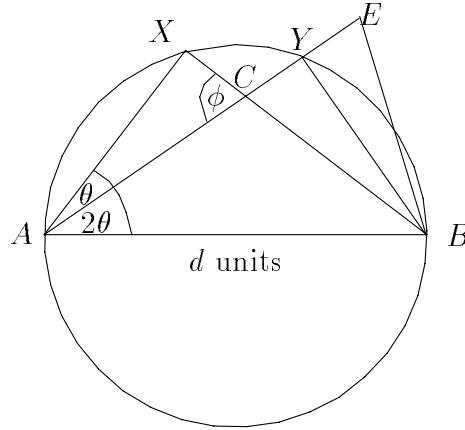


Figure 4: Picture for Question 3.

Second Solution. Let  $x = |AC|$ ,  $y = |CY|$ . Then

$$x \cos \theta = |AX| = d \cos \angle BAX = d \cos 3\theta,$$

and

$$x + y = |AY| = d \cos \angle BAY = d \cos 2\theta.$$

Hence

$$y = d\left[\cos 2\theta - \frac{\cos 3\theta}{\cos \theta}\right] = \frac{d(\cos 2\theta \cos \theta - \cos 3\theta)}{\cos \theta},$$

and so

$$\begin{aligned} |AE| &= x + 2y \\ &= \frac{d(2 \cos 2\theta \cos \theta - \cos 3\theta)}{\cos \theta} \\ &= \frac{d(2 \cos 2\theta \cos \theta - \cos 2\theta \cos \theta + \sin 2\theta \sin \theta)}{\cos \theta} \\ &= \frac{d(\cos 2\theta \cos \theta + \sin 2\theta \sin \theta)}{\cos \theta} \\ &= \frac{d \cos \theta}{\cos \theta} \\ &= d. \end{aligned}$$

# BCU MATHEMATICS CONTEST, MARCH 13, 1999

## SOLUTIONS FOR THE SPEED CONTEST PAPER FOR TEAMS

1. Let  $a$  be the number of initial participants. Then, initially, we have

$$\frac{55}{100} \cdot a$$

girls among them. We therefore obtain the equation

$$\frac{60}{100}(a + 5) = \frac{55}{100}a + 5$$

and therefore

$$a = 40.$$

So, initially, there were 40 participants and  $\frac{55 \cdot 40}{100} = 22$  girls among them.

2. For all pairs  $(x, y)$  with  $x^2 + y^2 = 1$  we obtain

$$f(x, y) = 2(x^2 + y^2) + y^2 = 2 + y^2.$$

Since the condition on  $x$  and  $y$  implies that  $0 \leq y^2 \leq 1$ , we have

$$2 \leq f(x, y) \leq 3.$$

We obtain the maximum value 3 for  $y = -1$  and  $y = 1$ , and the minimum value 2 for  $y = 0$ . The solutions therefore are:

Maxima:  $(0, -1)$ ,  $(0, 1)$

Minima:  $(1, 0)$ ,  $(-1, 0)$

3. Since  $1 \leq b < a$ , we have  $b \leq b^2 < a^2$  and so

$$b - a^2 < b - b^2 < b < a < 2a$$

which suffices.

OR

$$b - a^2 < b - b^2 \leq b - b = 0 < 1 \leq a < 2a.$$

4. The height  $h$  of the triangle is

$$h = \sqrt{a^2 - \frac{1}{4}a^2} = \frac{a}{2}\sqrt{3}.$$

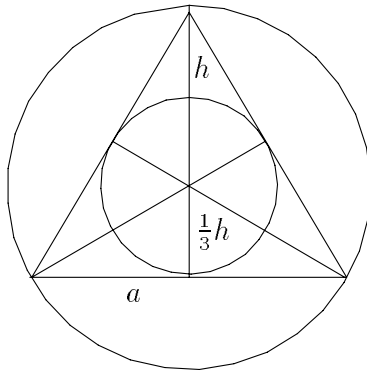


Figure 5: Question 4

Since the altitudes of the triangle intersect at a third of their lengths, and the centres of the circles coincide at the point of intersection of the altitudes, we get

$$r_i = \frac{a}{6}\sqrt{3}$$

as the radius of the inscribed circle, and

$$r_c = \frac{a}{3}\sqrt{3}$$

as the radius of the circumscribed circle.

5. For  $x = 0$  the recursion formula reduces to

$$p_{n+2}(0) = p_{n+1}(0).$$

Hence we obtain

$$\begin{aligned} p_1(0) &= 1, \\ p_2(0) &= 3, \\ p_3(0) &= 3, \\ p_5(0) &= p_4(0) = p_3(0) = 3. \end{aligned}$$

For  $x = -1$  the recursion formula is

$$p_{n+2}(-1) = p_{n+1}(-1) - 2p_n(-1),$$

and therefore

$$\begin{aligned} p_1(-1) &= 1, \\ p_2(-1) &= 2, \\ p_3(-1) &= 0, \\ p_4(-1) &= -4, \\ p_5(-1) &= -4. \end{aligned}$$

6. We obtain

$$\begin{aligned} 7^1 &= \dots 07 \\ 7^2 &= \dots 49 \\ 7^3 &= \dots 43 \\ 7^4 &= \dots 01 \\ 7^5 &= \dots 07 \\ 7^6 &= \dots 49 \\ &\vdots \\ 7^{76} &= \dots 01 \end{aligned}$$

since  $76 \bmod 4 = 0$ . Hence, the last two digits are 01.

7. Each of the 27 students taking three courses takes a course in Mathematics or Computer Science. Since 34 students take Spanish or History,  $45 - 34 = 11$  students take Mathematics or Computer Science, but neither of Spanish or History. Each of the latter group of students therefore attends a maximum of two courses. From this we obtain a lower bound of  $27 + 11 = 38$  students taking Mathematics or Computer Science.

We show that the bound is sharp, by giving a distribution of students among the courses such that exactly 38 students take Mathematics or Computer Science. It is as follows:

27 students take 3 courses.

11 students take Mathematics and Computer Science only.

7 students take Spanish and History only.

With this distribution, the assumptions of the question are satisfied and 38 students take Mathematics or Computer Science. Hence, the bound is sharp.