BCU Mathematics Contest 2000
Solutions

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Contest Web Page:
http://maths.ucc.ie/~pascal/verein/internat/index.html

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INDIVIDUAL CONTEST PAPER
Solutions

1 Note that x can not be 0. Multiplying by $|x|$ on both sides of the inequality yields

$$|3x - 2| < 2|x| - 1.$$

We have to distinguish three cases:

a) $x \geq \frac{2}{3}$: Then the inequality yields

$$3x - 2 < 2x - 1$$

from which we obtain that $x < 1$. So all x with $\frac{2}{3} \leq x < 1$ satisfy the inequality.

b) $0 \leq x \leq \frac{2}{3} = \frac{10}{15}$: Then the inequality yields

$$-3x + 2 < 2x - 1$$

and hence $x > \frac{3}{5} = \frac{9}{15}$. So all x with $\frac{3}{5} < x \leq \frac{2}{3}$ satisfy the inequality.

c) $x < 0$: Then the inequality yields

$$-3x + 2 < -2x - 1$$

from which we obtain $3 < x$ which is impossible.

Summing up the three cases, we obtain that exactly all x with

$$\frac{3}{5} < x < 1$$

satisfy the inequality.

2 Denote f by $f^{(0)}$. We obtain for all non-negative integers k:

$$f^{(n)} = 2^n \sin 2x \qquad \text{if } n = 4k \tag{1}$$
$$f^{(n)} = 2^n \cos 2x \qquad \text{if } n = 4k+1 \tag{2}$$
$$f^{(n)} = -2^n \sin 2x \qquad \text{if } n = 4k+2 \tag{3}$$
$$f^{(n)} = -2^n \cos 2x \qquad \text{if } n = 4k+3 \tag{4}$$

Intersections with the x-axis:

(1) $x = \frac{l\pi}{2}$ for all integers l
(2) $x = \frac{l\pi}{2} + \frac{\pi}{4}$ for all integers l
(3) $x = \frac{l\pi}{2}$ for all integers l
(4) $x = \frac{l\pi}{2} + \frac{\pi}{4}$ for all integers l

Maxima:

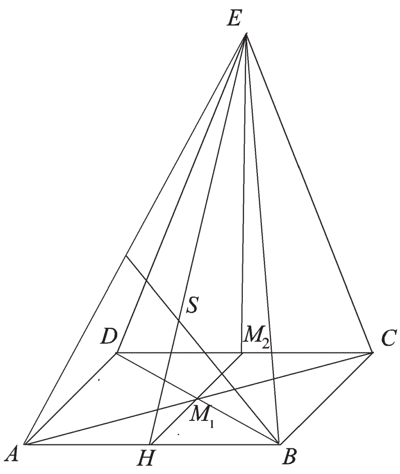
(1) $x = l\pi + \frac{\pi}{4}$ for all integers l
(2) $x = l\pi$ for all integers l
(3) $x = l\pi - \frac{\pi}{4}$ for all integers l
(4) $x = l\pi + \frac{\pi}{2}$ for all integers l

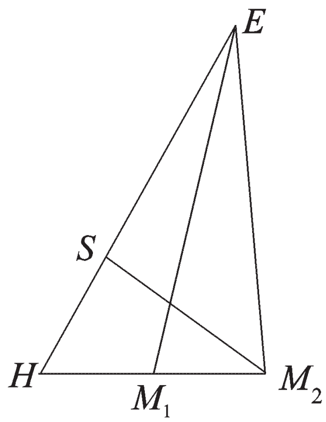
Minima:

(1) $x = l\pi - \frac{\pi}{4}$ for all integers l
(2) $x = l\pi + \frac{\pi}{2}$ for all integers l
(3) $x = l\pi + \frac{\pi}{4}$ for all integers l
(4) $x = l\pi$ for all integers l

Points of Inflection: Exactly the intersections with the x-axis above.

3





Let H be the mid-point of AB. Then HM_1M_2 are collinear. Since S is the centroid of $\triangle ABE$, HSE are collinear. So HM_2E form a triangle with M_1 on HM_2 and S on HE. Hence M_1E intersects SM_2.

4 (a) By taking $z = x$ in equation (2) we obtain

$$f(x, x) \leq 2f(x, y)$$

for all x, y, and since $f(x, x) = 0$ by (1), we obtain $f(x, y) \geq 0$.

(b) By taking $y = x$ in equation (2) we obtain

$$f(x, z) \leq f(x, x) + f(z, x) = f(z, x).$$

Since equation (2) holds for all x, y, z, we can also write it as

$$f(z, x) \leq f(z, y) + f(x, y),$$

and by taking $z = y$ we obtain

$$f(z, x) \leq f(z, z) + f(x, z) = f(x, z).$$

Hence $f(x, z) \leq f(z, x)$ and $f(z, x) \leq f(x, z)$ and so $f(z, x) = f(x, z)$.

An example of such a function would be the zero function $f(x, y) = 0$ for all x, y. Indeed, possible examples are exactly all pseudo-metrics on the real numbers.

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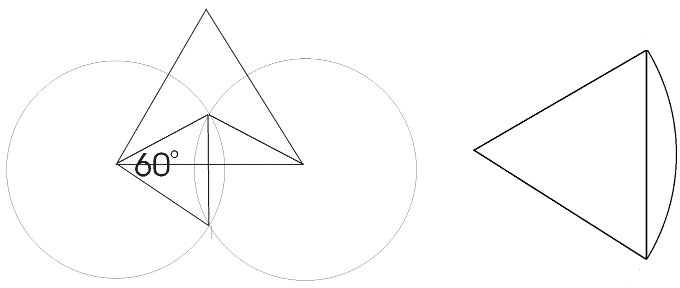
TEAM CONTEST PAPER
Solutions

1 The area of an equilateral triangle with side-length b is $\frac{\sqrt{3}}{4}b^2$.

The radius of one circle is $r = \frac{\sqrt{3}}{3}a$.

The area of one circle is $A_c = \frac{1}{3}\pi a^2$.

The area of the shamrock is $A = 3A_c - 3B$, where B is the area of the intersection between two circles.



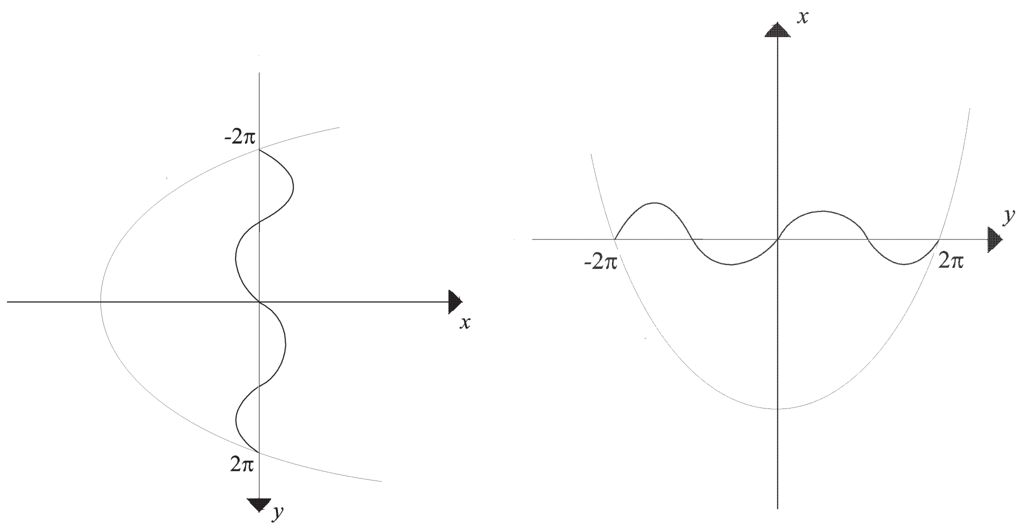
We can calculate

$$\begin{aligned} B &= 2\left(\frac{1}{6}A_c - \frac{\sqrt{3}}{4}r^2\right) \\ &= 2\left(\frac{1}{6}\cdot\frac{1}{3}\pi a^2 - \frac{\sqrt{3}}{4}\left(\frac{\sqrt{3}}{3}a\right)^2\right) \\ &= \left(\frac{\pi}{9} - \frac{\sqrt{3}}{6}\right)a^2 \\ &= \frac{2\pi - 3\cdot\sqrt{3}}{18}a^2. \end{aligned}$$

We finally obtain

$$A = 3A_c - 3B = \pi a^2 - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) a^2$$
$$= \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right) a^2$$

2



The easiest way to solve this is to write the curves as functions in y. The first two equalities then yield the function

$$f : y \mapsto y^2 - 4\pi^2$$

and the last equality the function

$$g : y \mapsto \sin y.$$

In order to calculate the intersection points, we have to solve the equality

$$\sin y = y^2 - 4\pi^2. \qquad (*)$$

By guessing, we obtain the two solutions $y = \pm 2\pi$.

We next show that $\pm 2\pi$ are the only solutions to the equality $(*)$. Indeed, we know that the slope of the sin function is at most 1. The slope of the function f at $y = \pm 2\pi$ is

$\pm 4\pi$. Also, $f'(y) < 1 \leq g'(y)$ for $y \leq -1$ and $f'(y) > 1 \geq g'(y)$ for $y \geq +1$. However, for $|y| \leq 1$ we have that $f(y) \leq 1 - 4\pi < -1$, so $\pm 2\pi$ are the only intersection points.

We finally have to solve the integral

$$\int_{-2\pi}^{2\pi} \sin y - y^2 + 4\pi^2 \mathrm{d}y$$

which yields

$$\begin{aligned}\left[\cos y - \frac{1}{3}y^3 + 4\pi^2 y\right]_{-2\pi}^{2\pi} &= 1 - \frac{1}{3}8\pi^3 + 8\pi^3 - \left(1 + \frac{1}{3}8\pi^3 - 8\pi^3\right) \\ &= \frac{4}{3}8\pi^3 = \frac{32}{3}\pi^3.\end{aligned}$$

3 (a) By trying, we find solutions for $n = 3, 5, 6, 8, 9, 10$. Now if there is a solution for some n, there obviously is a solution for all $n + 3k$, where k is a positive integer, since we can just add k groups of 3 members each. Since $8, 9, 10$ are three consecutive numbers, we obtain as an answer that it is possible for $n = 3, 5, 6$ and for all $n \geq 8$.

(b) We first determine the number $f(n)$ of groups as a function of n. This yields

$$\begin{aligned} f(n) &= \begin{cases} \frac{n}{3} & \text{if } n \text{ is divisible by } 3 \\ \frac{n-10}{3} + 2 & \text{if } n-1 \text{ is divisible by } 3 \text{ and } n \geq 10 \\ \frac{n-5}{3} + 1 & \text{if } n-2 \text{ is divisible by } 3 \end{cases} \\ &= \begin{cases} \frac{n}{3} & \text{if } n \text{ is divisible by } 3 \\ \frac{n-4}{3} & \text{if } n-1 \text{ is divisible by } 3 \text{ and } n \geq 10 \\ \frac{n-2}{3} & \text{if } n-2 \text{ is divisible by } 3 \end{cases}\end{aligned}$$

The function g which is to be determined is then obtained as

$$\begin{aligned} g(n) = \frac{n}{f(n)} &= \begin{cases} 3 & \text{if } n \text{ is divisible by } 3 \\ \frac{n}{\frac{n-10+6}{3}} & \text{if } n-1 \text{ is divisible by } 3 \text{ and } n \geq 10 \\ \frac{n}{\frac{n-5+3}{3}} & \text{if } n-2 \text{ is divisible by } 3 \end{cases} \\ &= \begin{cases} 3 & \text{if } n \text{ is divisible by } 3 \\ \frac{3n}{n-4} & \text{if } n-1 \text{ is divisible by } 3 \text{ and } n \geq 10 \\ \frac{3n}{n-2} & \text{if } n-2 \text{ is divisible by } 3 \end{cases}\end{aligned}$$

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SPEED CONTEST PAPER FOR TEAMS
Solutions

1 We obtain

$$\begin{aligned}
p_0(x) &= x^2 + 1 \\
p_1(x) &= x^2 \\
p_2(x) &= -x^2 - x^2 - 1 = -2x^2 - 1 \\
p_3(x) &= x^2 + x^2 + 1 - x^2 = x^2 + 1 \\
p_4(x) &= -x^2 - 1 + 2x^2 + 1 = x^2
\end{aligned}$$

and the process repeats. We obtain only three different polynomials.

2 Since the last digit of 1503985847 is a seven, a must be either 1, 3, 7, or 9. The number 1503985847 is not divisible by 3, but for $a = 3$ or 9, $934152a$ is divisible by 3. So a can not be 3 or 9. Trying $a = 1$ we obtain $1503985847 = 9341521 \cdot 161 + 966 = (9341521 + 6) \cdot 161$. Hence a must be 7.

3 The function is

$$f(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \end{cases}$$

(called McCarthy's 91 function). We obtain

$$\begin{aligned}
f(98) &= f(f(109)) \\
&= f(99) \\
&= f(f(110)) \\
&= f(100) \\
&= f(f(111)) \\
&= f(101) \\
&= 91.
\end{aligned}$$

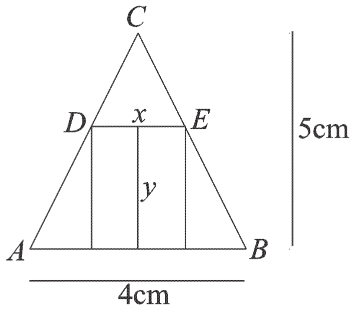
(Indeed the function is constant 91.)

4 Player B chooses $b = 0$ and $b' = 1$. We obtain the two equations

$$ax + y = c$$
$$a'y + z = c'$$

with the solution $x = 0$, $y = c$ and $z = c' - a'c$.

5 Let the length of DE be x and y be the distance between DE and AB.



Then we obtain

$$\frac{4}{x} = \frac{5}{5 - y}.$$

and this yields

$$y = 5 - \frac{5}{4}x.$$

We want to maximise the function f where

$$f(x) = 5x - \frac{5}{4}x^2.$$

We obtain

$$f'(x) = -\frac{5}{2}x + 5$$

which is zero for $x = 2$ only. Since $f''(2) = \frac{-5}{2}$, we obtain a maximum for $x = 2$, so DE is 2cm long.

6

$$x = y = z = \frac{\pi}{2}.$$

Because, in the first place, $1 \geq \sin x, \sin y, \sin z > 0$ and so the product $\sin x \sin y \sin z$ is strictly less than 1 if any factor is less than 1. Hence each factor equals 1.

7 We note that $p(x) = (x-\alpha)(x-\beta)(x-\gamma)$, hence

$$\begin{aligned} p(1) &= (1-\alpha)(1-\beta)(1-\gamma) \\ -p(-1) &= (1+\alpha)(1+\beta)(1+\gamma) \end{aligned}$$

and consequently

$$\frac{1-\alpha}{1+\alpha} \cdot \frac{1-\beta}{1+\beta} \cdot \frac{1-\gamma}{1+\gamma} = \frac{p(1)}{-p(-1)} = -1.$$