On the Relationship between Parsimonious Covering and Boolean Minimization *

Venu Dasigi and Krishnaprasad Thirunarayan

Department of Computer Science and Engineering
Wright State University Research Center,
Dayton, OHIO 45420.
email: vdasigi@cs.wright.edu, tkprasad@cs.wright.edu

Abstract

Minimization of Boolean switching functions is a basic problem in the design of logic circuits. The designer first comes up with a switching function expressed in terms of several binary input variables that satisfies the desired functionality, and then attempts to minimize the function as a sum of products or product of sums. It turns out that a sum of products form of a switching function that has no redundancy is a union of prime implicants of the function.

In this paper we would like to explicate some of the relationships of the boolean minimization problem to a formalization of abductive inference called parsimonious covering. Abductive inference often occurs in diagnostic problems such as finding the causes of circuit faults [Reiter, 87] or determining the diseases causing the symptoms reported by a patient [Peng and Reggia, 90]. Parsimonious covering involves covering or accounting for the set of observed manifestations using a parsimonious set of explanations that can account for the observations. The relationship of parsimonious covering to boolean minimization has been noted by the developers of the theory; we intend to pursue a detailed mapping here.

1 Introduction

In Boolean minimization, one is interested in minimizing the number of terms (possibly with fewer literals) in the expression of a Boolean switching function [Kohavi, 78]. In other words, the goal is to find an expression for a minimal cost logic circuit that causes the same functionality as the given boolean function. This goal is very similar to the abductive goals of explaining the faults in circuits or diagnosing a set of medical symptoms in a patient. One of the computer models for certain classes of abductive inference is parsimonious covering. This theory has been developed for diagnostic problem solving [Peng and Reggia, 90], but has also been extended to other domains such as language processing [Dasigi, 89]. Parsimonious covering involves covering or accounting for the set of observed manifestations using a parsimonious set of possi-
ble causes. Once again, this is very similar to covering the desired switching function using a minimal or irredundant set of terms as is the goal of Boolean minimization. Not surprisingly, minimality and irredundancy have been used as criteria of parsimony in parsimonious covering theory. Several of these relationships have been noted by the developers of parsimonious covering, but have never been explicated before, to our knowledge. We pursue a detailed mapping here, and observe that features of either problem may be captured in terms of those of the other. Thus, we hope this paper will be of interest to either community. After quick reviews of Boolean minimization and parsimonious covering, we show how they capture each other’s features, and conclude with our plans for further work.

2 Boolean Minimization Problem

A brief review of the Boolean Minimization problem (henceforth referred to as BMP) follows [Kohavi, 78]. The boolean constants true and false are denoted as 0 and 1; the boolean operations and, or and not as \( \land \), \( \lor \), and \( \neg \); and boolean variables as \( x_0, x_1, \ldots, x_n \). Any boolean constant or variable is a boolean expression, and if \( B_1 \) and \( B_2 \) are boolean expressions, then so are \( \neg B_1, \neg B_2, B_1 \lor B_2 \) and \( B_1 \land B_2 \). \( x \) and \( \neg x \) are called literals. A conjunction (respectively, disjunction) of literals is called a product term (resp. sum term). A product term (resp. sum term) containing literals involving all input variables is called a minterm (resp. maxterm). A boolean expression is in sum-of-products form (resp. product-of-sums form) if it is expressed as a disjunction (resp. conjunction) of product terms (resp. sum terms). Two boolean expressions \( B_1 \) and \( B_2 \) containing variables \( \{x_0, x_1, \ldots, x_n\} \) are said to be logically equivalent if \( B_1 \) and \( B_2 \) have the same values for all possible combinations of values of the variables \( \{x_0, x_1, \ldots, x_n\} \). Corresponding to every boolean expression there exist logically equivalent boolean expressions that are in sum-of-products (resp. product-of-sums) form.

A sum-of-products expression is minimal if there is no other expression with smaller number of product terms and with fewer literals. A sum-of-products expression is irredundant if it is not possible to delete a product term or a literal from it without altering its logical value. A minimal expression is not always unique, but is always irredundant. However, an irredundant expression may not necessarily be minimal.

An implicant of a boolean expression is a product term that logically implies it. A prime implicant of a boolean expression is an implicant that does not logically imply any other implicant of the expression. An essential prime implicant is a prime implicant that does not logically imply any disjunction of other prime implicants.

A boolean expression \( f(x_0, x_1, \ldots, x_n) \) represents a monotonic function, if it satisfies the following condition:

\[
\forall i \leq n : [f(x_0, \ldots, x_i := 0, \ldots, x_n) = 1] \Rightarrow [f(x_0, \ldots, x_i := 1, \ldots, x_n) = 1]
\]

3 Parsimonious Covering Theory

A brief review of Parsimonious Covering Problem (henceforth referred to as PCP) follows [Peng and Reggia, 90]. A diagnostic
Figure 1: Parsimonious Covering Problem

problem P is a 4-tuple \( (D, M, C, M^+) \) where \( D \) is a finite set of disorders; \( M \) is a finite set of manifestations; \( C \subseteq D \times M \) is the causation relation; and \( M^+ \subseteq M \) is the set of observed manifestations. For any \( d_i \in D \) and \( m_j \in M \), effects\( (d_i) = \{m_j \mid (d_i, m_j) \in C\} \) and causes\( (m_j) = \{d_i \mid (d_i, m_j) \in C\} \). For any \( D_I \subseteq D \), effects\( (D_I) = \{m_j \mid (d_i, m_j) \in C \wedge d_i \in D_I\} \). The set \( D_I \subseteq D \) is said to be a cover of \( M_I \subseteq M \) if \( M_I \subseteq \text{effects}(D_I) \).

A set \( E \subseteq D \) is said to be an explanation of \( M^+ \) for a problem \( P = (D, M, C, M^+) \) iff \( E \) covers \( M^+ \) and \( E \) satisfies a given parsimony criterion. A cover \( D_I \) of \( M_J \) is said to be minimal if its cardinality is smallest among all covers of \( M_J \). A cover \( D_I \) of \( M_J \) is said to be irredundant if none of its proper subsets is also a cover of \( M_J \). A minimal cover is irredundant, but the converse does not hold.

We illustrate the above definitions with an example depicted in Figure 1. \{\( d_1, d_2, d_3 \)\} are the diseases, \{\( m_1, m_2, m_3 \)\} are the manifestations. \{\( d_1, d_2 \)\} is a cover of \{\( m_1, m_2 \)\}, while \{\( d_4 \)\} is the minimum cover of \{\( m_1, m_2 \)\}. \{\( d_1, d_3 \)\} and \{\( d_2, d_4 \)\} are two minimal covers of \{\( m_1, m_2, m_3 \)\}. \{\( d_3 \)\} and \{\( d_2, d_4 \)\} are both irredundant covers of \{\( m_2, m_3 \)\}; the former one is minimal, while the latter one is not. Algorithms for computing minimal and irredundant covers have been developed and have been extended to more complex knowledge structures involving chains of causal links (e.g., an overheated resistor may cause a nearby transistor to malfunction, which, in turn, may change the output of a gate).

4 Encoding a PCP as a BMP

We now show that an instance of PCP can be encoded as an instance of BMP. The set of diseases of PCP is the set of boolean variables of BMP. Covering a manifestation \( m \) requires one of the \( \text{causes}(m) \) to be present. This is equivalent to saying that the disjunction of \( d \)’s in \( \text{causes}(m) \) be true. To cover \( M^+ \), the set of observed manifestations, we need to cover each manifestation in \( M^+ \). Thus, the boolean expression obtained by conjoining the aforementioned disjunctions (of causative diseases) for each observed manifestation should be true. The following result specifies the relationship between PCP and its encoding as a BMP.

Lemma 1 Given an instance of the diagnostic problem \( P = (D, M, C, M^+) \), the corresponding instance of the boolean minimization is the product-of-sums expression

\[
B_{ps} = \bigwedge \{\neg \overline{d_i} \mid (d_i, m_j) \in C \wedge m_j \in M^+\},
\]

where the set \( D \) corresponds to the set of boolean variables. Let \( B_{ps} \) be the corresponding sum-of-products expression obtained by distributing \( \neg \)s over \( \wedge \)s. Then, every implicant of \( B_{ps} \) corresponds to a cover of \( M^+ \); every prime implicant of \( B_{ps} \) to an
irredundant cover of $M^+$; and every prime implicant $B_i$, with the least number of literals, to a minimal cover of $M^+$.

We illustrate this with the example shown in Figure 1. If $M^+ = \{m_2, m_3\}$, then the PCP instance can be encoded as: $(d_2 \lor d_3) \land (d_3 \lor d_4)$. A logically equivalent sum-of-products form is:

$$d_2 \land d_3 \lor d_2 \land d_4 \lor d_3 \lor d_3 \land d_4.$$ 

The prime implicants are $d_3$ and $(d_2 \land d_4)$. Only $d_3$ qualifies to be minimal (and hence, irredundant), while $d_2 \land d_4$ is irredundant (but not minimal).

In general, minimal expressions are not unique. For instance, both

$$\neg x_1 \lor \neg x_3 \lor x_1 \land \neg x_2 \lor x_2 \land x_3$$

$$\neg x_1 \lor x_2 \lor \neg x_2 \land \neg x_3 \lor x_1 \land x_3$$

are logically equivalent distinct minimal expressions. On the other hand, observe that the Boolean expressions encoding a PCP does not contain the negation operator. Thus, it can be shown that

Lemma 2 The boolean expression arising as the encoding of the PCP represents a monotonic function, which can be represented by a unique minimal expression in sum-of-products form.

This lemma is the basis of the following algorithm to compute all irredundant and minimal covers. Given an instance of a PCP, encode it as a BMP, and transform the expression so obtained into a logically equivalent expression in sum-of-products form by distributing $\land$ over $\lor$. Each disjunct is an implicant of the original expression. The set of implicans so obtained is then partitioned into groups $g_1, g_2, \ldots, g_n$, where $g_i$ contains all implicants with $i$ distinct literals. Starting from group $g_2$, delete implicants, from all groups $g_i$, that logically imply other implicants that appear in groups with lower index. This procedure terminates leaving only prime implicants, which constitute the set of irredundant covers. The members of the nonempty group $g_i$ with the least index $i$ constitute the set of minimal covers.

5 Encoding a BMP as a PCP

Parsimonious covering was originally conceived as a formal model of the way diagnostic inferences are performed by human diagnosticians. The first version of the theory was a generalization of the set covering problem of mathematics [Edwards, 62]. Parsimonious covering has found applications in error classification in some discrete sequential processes [Ahuja, 85], software engineering [Basili and Ramsey, 85], growth models of biological tree structures [Tagamets and Reggia, 85], treatment selection [Neapolitan, et al., 87] and natural language processing [Dasigi, 89]. From this perspective, it is interesting to see yet another application in Boolean minimization for parsimonious covering.

The problem of minimizing a Boolean expression essentially consists of two major steps: that of determining all prime implicans of the function and that of selecting a minimal (or irredundant) subset of prime implicans that can cover all the minterms of the given boolean function. Now, it is obvious that there exists a straightforward mapping between concepts underlying parsimonious covering (especially in the context of diagnostic problems) and those un-
Boolean Minimization | Parsimonious Covering
--- | ---
Minterms | Manifestations
Prime Implicants | Disorders
Implication | Causal Relation
$x$ is implied by $x \land \neg y$ | $d$ causes $m$
$x$ covers $x \land \neg y$ | $d$ covers $m$
Boolean function to be minimized | Observed manifestations
Minimal form of boolean function | Minimal cover
Irredundant set of prime implicants | Irredundant cover
don't cares | Manifestations that may or may not be covered
essential prime implicants | Disorders covering pathognomonic manifestations

Table 1: Mapping between Parsimonious Covering and Boolean Minimization

From this preliminary work, we draw the following conclusions, which also indicate directions for further work:

- It may be noted that only the prime implicants of a given boolean function in a BMP, rather than any general product terms, are considered analogous to disorders in a PCP. If any general product term were treated as the BMP-analog of a disorder, then there would be a trivial minimum cover, namely, 1, that can “cover” (in a different sense) any boolean function. The specific choice of prime implicants as the BMP-analog of disorders captures the important notion of logical equivalence of the minimal cover to the original boolean expression.

- As already mentioned, BMP consists of two major steps. An interesting observation in this context is that in Section 4, a PCP has been mapped into the first step of a BMP (the determination of all prime implicants), while in Section 5, the second step of a BMP...
the selection of a minimal or irredundant subset of prime implicants that can cover the given boolean function) has been mapped into a PCP. This suggests the possibility that the complete BMP may be equivalent to the PCP. After all, parsimony (lack of redundancy) is a notion germane to prime implicants.

- It does seem possible to start with a boolean function expressed as the sum of several minterms (product terms involving all input variables) and cover it with product terms involving one fewer variable that are implicants of the function, and then with still smaller product terms, etc. Each step of covering in this process roughly corresponds to what Peng and Reggia call layers (more precisely, pseudo-layers) [Peng and Reggia, 90]. We hope that work in this direction identifies closer ties between the two formalisms and also that theoretical results in each area translate into equivalent results in the other. In particular, it would be interesting to see if transitivity of complete sets of irredundant covers applies to analogous notions in the BMP.

References


