On the Relationship between Abductive Reasoning and Boolean Minimization

Venu Dasigi,
Department of Computer Science,
Southern Polytechnic State University,
Marietta, GA 30060-2896.
vdasigi@spsu.edu, (770)-528-5559

Krishnaprasad Thirunarayan,
Department of Computer Science and Engineering,
Wright State University, Dayton, OH 45435.
tkprasad@cs.wright.edu, (937)-775-5109

Abstract

Abductive reasoning involves determining a parsimonious set of explanations that can account for a set of observations. In the Boolean minimization problem, the designer attempts to express a Boolean formula as a *sum of products* or *product of sums* expression of the smallest size that satisfies the desired function. In this paper, we show that independent abduction problem can be encoded as an instance of Boolean minimization problem, and conversely, a Boolean minimization problem as an abduction problem. We then consider the application of the transitivity results from the parsimonious covering theory to the Boolean minimization. We conclude with a brief comparison to the related work.

1 Introduction

Abductive reasoning involves covering or accounting for a set of observed manifestations using a parsimonious set of possible causes that can explain them. This is very similar to covering a desired switching function using a minimal or irredundant set of terms which is the goal of Boolean minimization. Not surprisingly, *minimality* and *irredundancy* have been used as criteria of parsimony in abductive reasoning. The relationship between boolean minimization and abductive reasoning have been noted by the developers of various theories of abductive reasoning, but have never been explicated before, to our knowledge [1, 4, 15, 16, 20, 24, 25, 26]. We pursue a detailed mapping here, and observe that features of either problem may be captured in terms of those of the other. After quick reviews of Boolean minimization and models of abductive reasoning, we show how they capture each other's features, and suggest that results from one area may well be extendable into the other. This qualitative account provides a better understanding of the similarities and the differences between the two problems.

2 Boolean Minimization Problem

A brief review of the Boolean Minimization problem (henceforth referred to as BMP) follows (using the notation given in [16]): The Boolean *constants* true and false are denoted as 1 and 0; the Boolean operations and, or and not as $\wedge$, $\vee$ and $\neg$; and Boolean *variables* as $x_0, x_1, \ldots, x_n$. Any Boolean constant or variable is a *Boolean expression*, and if B_1 and B_2 are Boolean expressions, then so are $\neg B_1$, $\neg B_2$, $B_1 \vee B_2$ and $B_1 \wedge B_2$. x and $\neg x$ are called *literals*. A conjunction (respectively, disjunction) of literals is called a *product term* (resp. *sum term*). A product term (resp. sum term) containing literals involving *all* input variables is called a *minterm* (resp. *maxterm*). A Boolean expression is in *sum-of-products form* [or *disjunctive normal form*] (resp. *product-of-sums form* [or *conjunctive normal form*]) if it is expressed as a disjunction (resp. conjunction) of product terms (resp. sum terms). Two Boolean expressions B_1 and B_2 containing variables $\{x_0, x_1, \ldots, x_n\}$ are said to be *logically equivalent* if B_1 and B_2 have the same values for all possible combinations of values of the variables $\{x_0, x_1, \ldots, x_n\}$. Corresponding to every Boolean expression there exist logically equivalent Boolean expressions that are in sum-of-products (resp. product-of-sums) form.

A sum-of-products expression is *minimal* if there is no other equivalent expression with smaller number of product terms and with fewer literals. A sum-of-products expression is *irredundant* if it is not possible to delete a product term or a literal from it without changing its logical value. A minimal expression is not always unique, but is always irredundant. However, an irredundant expression may not be minimal.

An *implicant* of a Boolean expression is a product term that logically implies it. A *prime implicant* of a Boolean expression is an implicant that does not logically imply any other (non-logically equivalent) implicant of the expression. An *essential prime implicant* is a prime implicant that does not logically imply any disjunction of other prime implicants.

Often visual representations such as Karnaugh maps, Quine-McCluskey tabulation and the prime implicant chart method are used for determining a minimal form for a Boolean expression [15, 16]. See [4] for details on the history of logic minimization and an efficient implementation of logic minimization algorithms.

To represent a Boolean expression of n variables, a Karnaugh map of 2^n cells is used, where each cell represents a minterm. Such a map is usually shown as a square grid of $2^{n/2}$ by $2^{n/2}$ cells if n is even or as a rectanguar grid of $2^{(n-1)/2}$ by $2^{(n+1)/2}$ cells when n is odd. The expression itself is shown in the map by marking the cells corresponding to the minterms in the expression. The idea is that "adjacent" cells can be combined into so-called *subcubes*, which correspond to implicants with fewer literals than the terms being combined. A subcube of order m is a collection of 2^m cells, in which each cell is adjacent to m other cells. In a Karnaugh map of n variables, a subcube of order m corresponds to an implicant containing $n - m$ literals. In this paper, we use the phrases, "subcubes of order i" and "implicants of order i" interchangeably.

A Boolean expression $f(x_0, x_1, \ldots, x_n)$ represents a *monotonic* function, if it satisfies the following condition:

$$\forall i \leq n: \ \ [f(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \ = \ 1]$$

$$\Rightarrow \ [f(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) \ = \ 1]$$

That is, if the value of a monotonic function is 1 for a certain input, then the value cannot be changed to 0 by changing an input bit from 0 to 1. (In [4], these functions are referred to as *monotone increasing* and are included in *unate* functions.)

3 Models of Abductive Reasoning

We review briefly the definition of the abduction problem (henceforth referred to as AP), following the exposition in [1]. An *abduction problem* P is a 4-tuple $\langle D, M, C, M^+ \rangle$ where D is a finite set of disorders; M is a finite set of manifestations; $C : \mathcal{P}(D) \longrightarrow \mathcal{P}(M)$ is the causation function; and $M^+ \subseteq M$ is the set of observed manifestations. ($\mathcal{P}(S)$ represents the powerset of S, and C maps a set of disorders to the corresponding set of manifestations.) For any set of disorders $D_i \in \mathcal{P}(D)$ and $m_j \in M$, $effects(D_i) = C(D_i)$ and $causes(m_j) = \{D_i \mid m_j \in C(D_i)\}$. ($effects$ is the same as C.) The set $D_I \subseteq D$ is said to be a *cover* of $M_J \subseteq M$ if $M_J \subseteq effects(D_I)$.

A set $E \subseteq D$ is said to be an *explanation* of M^+ for a problem $P = \langle D, M, C, M^+ \rangle$ iff E covers M^+ and E satisfies a given "miminality" (parsimony) criterion. A cover D_I of M_J is said to be *minimal* if its cardinality is smallest among all covers of M_J. A cover D_I of M_J is said to be *irredundant* if none of its proper subsets is also a cover of M_J. A minimal cover is irredundant, but the converse does not hold. One can consider different classes of abduction problems based on the additional properties satisfied by the causation function C, as described below [1].

- *Independent Problems*: An abduction problem is *independent* if:

$$\forall D_I \subseteq D: \; C(D_I) \;=\; \bigcup_{d \in D_I} C(\{d\}).$$

In other words, the different disorders cause disjoint symptoms.

- *Monotonic Problems*: An abduction problem is *monotonic* if:

$$\forall D_I \subseteq D: \; C(D_I) \;\supseteq\; \bigcup_{d \in D_I} C(\{d\}).$$

In other words, it is possible for a combination of disorders to cause additional symptoms over and above those caused by the individual disorders.

- *Incompatibility Problems*: In *incompatibility* abduction problems, it is possible to represent the fact that a set of disorders is mutually exclusive. Thus, an abduction problem is an incompatible problem if:

$$\forall D_I \subseteq D: \; [\exists E \subset D_I : \; C(E) \;=\; \emptyset] \;\Rightarrow\; [C(D_I) \;=\; \emptyset].$$

Example 1 We illustrate the above definitions with an independent problem depicted in Figure 1. (Only a part of the causation function that corresponds to its value on the singleton sets has been shown. The rest of it can be determined from this information using the fact that it is an independent problem.) $D = \{d_1, d_2, d_3, d_4\}$ is the set of all disorders, $M = \{m_1, m_2, m_3\}$ is the set of all possible manifestations. $\{d_1, d_2\}$ is a cover of $\{m_1, m_2\}$, while $\{d_2\}$ is the minimum cover of $\{m_1, m_2\}$. $\{d_1, d_3\}$ and $\{d_2, d_4\}$ are two minimal covers of $\{m_1, m_2, m_3\}$. $\{d_3\}$ and $\{d_2, d_4\}$ are both irredundant covers of $\{m_2, m_3\}$; the former one is minimal, while the latter one is not.

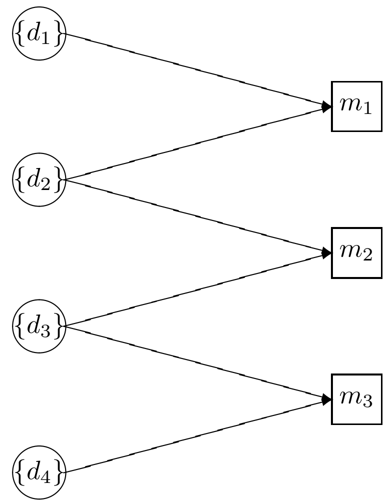

Figure 1: Parsimonious Covering Problem

4 Encoding an AP as a BMP

We now show that an instance of AP can be encoded as an instance of BMP. The set of disorders of AP is the set of Boolean variables of BMP. Covering a manifestation m requires one of the members of $causes(m)$ to be present. This is equivalent to saying that the disjunction of D_I's in $causes(m)$ be true, where each D_I is a conjunction of disorders $d \in D_I$. Recall that each member of $causes(m)$ is a set of disorders. To cover M^+, the set of observed manifestations, we need to cover each manifestation in M^+. Thus, the Boolean expression obtained by conjoining the aforementioned disjunctions (of causative disorders) for each observed manifestation should be true. But notice that in the case of incompatibility problem, we need to ensure that the combinations of mutually exclusive disorders are considered as invalid explanations.

Given an instance of the abduction problem $P = \langle D, M, C, M^+ \rangle$, we can form the corresponding instance of the Boolean minimization B as follows: Let the Boolean expression B_{ps} be

$$\bigwedge\{\bigvee\{\wedge_{d \in D} d \mid m \in C(D)\} \mid m \in M^+\},$$

where the set D corresponds to the set of Boolean variables and $\mid$ stands for "*such that*". Recall that B_{ps} can be obtained by taking a conjunction over the observed manifestations of the disjunctions of the possible causes of each manifestation. Each such cause can itself be a conjunction of disorders in the general case. B_{ps} is in the standard product-of-sums expression [aka conjunctive normal form] for independent problems. This assumes that no other sets of disorders are capable of causing M^+, and that the manifestations in M^+ cannot occur without being caused. Let B_{sp} be the corresponding sum-of-products expression [aka disjunctive normal form] obtained by distributing $\wedge$s over $\vee$s in B_{ps}. For an arbitrary Boolean expression like

$$(x_1 \vee x_2) \ \wedge \ (x_2 \vee x_3) \ \wedge \ \ldots \ \ \wedge \ (x_{n-1} \vee x_n),$$

performing this conversion can have exponential complexity in terms of the number of Boolean variables involved. We delete product terms from B_{sp} that contain incompatible disorders to obtain the corresponding instance of the Boolean minimization B.

The following result specifies the relationship between AP and its encoding as a BMP. Reiter has proved a similar result as Theorem 7.1 in [24] and as Theorem 3.13 in [9].

Theorem 1 *Let B be the instance of BMP corresponding to $P = \langle D, M, C, M^+\rangle$. Then, every implicant of B corresponds to a cover of M^+; every prime implicant of B to an irredundant cover of M^+; and every prime implicant B_{sp} with the least number of literals, to a minimal cover of M^+.*

Proof: The Boolean expression B_{ps} is a product of possible causes for the manifestations in M^+. Thus, each product term of B_{sp} covers all of M^+. Thus, every implicant of B_{sp} corresponds to a cover of M^+. From the definition of prime implicant, it follows that prime implicants form irredundant covers of M^+, and prime implicants with least number of literals form the minimal covers. Furthermore, essential prime implicants are disorders covering pathognomonic manifestations. •

In general, a Boolean expression can have several minimal sum-of-products representation. For example, the following three distinct minimal expressions are equivalent.

$$(x_1 \wedge x_3) \vee (x_1 \wedge \neg x_2) \vee (\neg x_2 \wedge \neg x_3 \wedge x_4) \vee (\neg x_1 \wedge x_2 \wedge x_4)$$

$$(x_1 \wedge x_3) \vee (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge \neg x_3 \wedge x_4) \vee (\neg x_1 \wedge x_2 \wedge x_4)$$

$$(x_1 \wedge x_3) \vee (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge \neg x_3 \wedge x_4) \vee (\neg x_2 \wedge x_3 \wedge x_4)$$

However, notice that the Boolean expressions encoding an AP does not contain the negation operator. In fact, it can be shown that

Theorem 2 *The Boolean expression B arising as the encoding of an AP represents a* monotonic *function which can be represented by a* unique *minimal expression B_{sp} in sum-of-products form.*

Proof: Given a Boolean expression B for a monotonic function, we can convert it into an equivalent minimal expression in sum-of-products form B_{sp} which contains only prime implicants. Note that such boolean expressions do not contain any negation. We wish to show that this expression is unique. We proceed by contradiction.

Assume that there exist two distinct minimal Boolean expressions X and Y —

$$(x_{11} \wedge \ldots \wedge x_{1n_1}) \ \vee \ \ldots \ \vee \ (x_{m1} \wedge \ldots \wedge x_{mn_m})$$

$$(y_{11} \wedge \ldots \wedge y_{1j_1}) \ \vee \ \ldots \ \vee \ (y_{i1} \wedge \ldots \wedge y_{ij_i})$$

and they represent the same Boolean function. Also, assume that $(n_1 \leq \ldots \leq n_m) \ \wedge \ (j_1 \leq \ldots \leq j_i)$. Because of the minimality assumption on terms and the monotonicity assumption on the function the expression represents, assigning 1, i.e., true to variables in $\{x_{11}, \ldots, x_{1n_1}\}$ and 0, i.e., false to all other variables makes all but the first term in X false. The same truth assignment must satisfy Y because Y is equivalent to X. Firstly, we note that if Y evaluates to true it must be the case that one of the product terms must evaluate to true. By the definition of the truth assignment, this product term cannot be an "extension" of $x_{11} \wedge \ldots \wedge x_{1n_1}$ because then it will be false. If Y has a product term that is included in $x_{11} \wedge \ldots \wedge x_{1n_1}$, then we can exchange X and Y, and repeat the aforementioned argument. Thus, it must be the case that both X and Y contain $x_{11} \wedge \ldots \wedge x_{1n_1}$, as all other possible product terms will evaluate to false. This argument can be repeated on the remaining portion of the two expressions to prove the uniqueness of the minimal form. •

Also note that [4] proves a similar result for *unate* functions.

The above result is the basis for the *completeness* of the following algorithm for computing irredundant and minimal covers. In other words, the uniqueness of the minimal expression implies that determining the minimum Boolean expression is sufficient for computing *all* irredundant and minimal covers of the manifestations. Given an instance of an AP, encode it as an instance of BMP as described. Each disjunct in the *sum-of-products* expression is an implicant of the original expression. (We assume that all the variables in the implicant term are distinct.) The set of implicants so obtained is then partitioned into groups $g_1, g_2, \ldots, g_n$, where g_i contains all implicants with

i distinct literals. Starting from group g_2, delete implicants, from all groups g_i, that logically imply other implicants that appear in groups with lower index. This procedure terminates leaving only prime implicants, which constitute the set of irredundant covers. The members of the nonempty group g_i with the least index i constitute the set of minimal covers.

Example 2 We illustrate this algorithm on the abduction problem given in Figure 1. If $M^+ = \{m_2, m_3\}$, then the problem instance can be encoded as: $(d_2 \vee d_3) \wedge (d_3 \vee d_4)$. A logically equivalent sum-of-products form is:

$$d_2 \wedge d_3 \ \vee \ d_2 \wedge d_4 \ \vee \ d_3 \ \vee \ d_3 \wedge d_4.$$

The implicants in the above expression can be partitioned into two groups: $g_1 = \{d_3\}$ and $g_2 = \{\{d_2, d_3\}, \{d_2, d_4\}, \{d_3, d_4\}\}$. The result of deleting terms in g_2 containing terms in g_1 is $\{\{d_2, d_4\}\}$. Thus, the prime implicants of the expression are d_3 and $(d_2 \wedge d_4)$. Only d_3, which belongs to the group with the least index (that is, g_1), qualifies to be minimal (and hence, irredundant), while $d_2 \wedge d_4$, which belongs to g_2, is irredundant (but not minimal).

An incremental algorithm to compute the set of minimal covers is given in [17]. According to [2] [5], the problem of finding an irredundant cover is tractable, while the problem of computing minimal covers is not.

5 Viewing a BMP as an AP

In this section, we will also refer to *parsimonious covering theory* (PCT), a specialized, formal model of the way inferences are performed by human diagnosticians. The first version of the theory was a generalization of the set covering problem of mathematics [10]. Parsimonious covering has been applied to a variety of tasks, such as, natural language processing [8] and software engineering [3].

Parsimonious covering theory models independent abduction problems in great detail. So, as pointed out at the end of Section 3 and as indicated in Figure 1, it is sufficient to think of the domain of the causation function C as just the set D, rather than the powerset $\mathcal{P}(D)$. The image of any non-singleton subset of D may be determined by applying the definition of independent problems given in Section 3.

Further, the PCT deals with more complex knowledge structures than the bipartite one indicated in Figure 1. In a bipartite problem structure, there is no causal chaining, that is, disorders directly cause manifestations; thus, there are just two *layers* of entities connected by causal links. Peng and Reggia describe problems where the underlying knowledge involves causal chaining, where a disorder may cause a manifestation indirectly through one or more layers of intermediate syndromes. Some interesting properties of irredundant covers across such layered problems were proved, and algorithms developed to compute irredundant covers, exploiting such properties [20]. Such knowledge structures occur in various task domains, ranging from diagnostic problems to language understanding, where phrase sense structures may be built up from individual word senses, and also contribute to the overall interpretation of a sentence. We believe that Boolean minimization can also be formulated in a layered structure.

The problem of minimizing a Boolean expression consists of two major steps: that of determining all prime implicants of the function and that of selecting a minimal (or irredundant) subset of prime implicants that is equivalent to all the minterms of the given Boolean function. In Section 4, an AP has been mapped into the first step of a BMP (the determination of all prime implicants). Below, we map the second step (the selection of a minimal or irredundant subset of prime implicants that can cover the given Boolean function) into PCT, a model of a particular kind of AP. This mapping is not surprising because both PCT and BMP involve notions of covering and parsimony. We summarize several such relationships in Table 1.

In a BMP, a set of minterms represents a Boolean expression to be minimized, analogous to a set of observed manifestations to be explained in an AP. The minimized form of a Boolean expression always consists of prime implicants, just as disorders are used to explain manifestations. A prime implicant is said to cover all the minterms that imply it, which is analogous to a disorder's potential to cover all the manifestations it can cause. In a BMP, a prime implicant may be implied by don't-care minterms. Similarly, in a specific AP, a disorder need not cause all manifestations it can po-

Boolean Minimization	Parsimonious Covering
minterms	manifestations
prime implicants	disorders
subcubes/implicants [16]	intermediate syndromes
implication	causal relation
Boolean function to be minimized	observed manifestations
minimized form of Boolean function	minimal cover
irredundant set of prime implicants	irredudant cover
don't cares	manifestations that may or may not be covered
essential prime implicants	disorders covering pathognomonic manifestations

Table 1: Mapping between Parsimonious Covering and Boolean Minimization

tentially cause. Any minimized form of a switching function *must* consist of essential prime implicants. The analogous concept in an AP would be disorders that uniquely correspond to pathognomonic manifestations. Finally, in a typical BMP, only the minimized form of a Boolean function may be of interest, while in an AP, one can talk about minimal covers as well as irredundant covers. In diagnostic problems, the latter type of covers appear to be so interesting that they are often called (syntactically) minimal covers.

5.1 A Closer Look at the Mapping

Given that an AP may be viewed as a BMP, as in Section 4, and that a BMP may be viewed as a particular kind of AP, as indicated above, the possibility that the complete BMP may be equivalent to the AP arises. After all, parsimony (lack of redundancy) is a notion germane to prime implicants.

It does seem possible to start with a Boolean function expressed as the sum of several minterms (product terms involving *all* input variables) and cover it with product terms involving one fewer variable that are implicants of the function, and then with still smaller product terms, etc. In the Boolean minimization terminology introduced in Section 2, minterms (which correspond to implicants of order 0) that are adjacent to each other may be covered by an implicant of order 1; two "adjacent" implicants of order 1 can be covered by an implicant of order 2, etc. It is important, however, to recognize that this notion of covering in the BMP context also requires that the covering subcube be *logically equivalent* to the covered subcubes, and any algorithm that attempts to derive such covers in the BMP must take this all-important constraint into account. (In the following material on Boolean minimization when the word "cover" is used as a noun or as a verb, it is tacitly requires logical equivalence.) In the parsimonious covering framework, a disorder may cause, in a given case, only a few (but not necessarily all) of the symptoms it is capable of causing, but the above constraint in the BMP means that in order to "hypothesize" a higher order subcube, all the order 0 subcubes (minterms) it is capable of causing must be present in the Boolean expression being minimized; if this logical equivalence is not captured, the higher order subcube would not correspond to an implicant. (We will show below an inexpensive way of taking care of this unavoidable issue.) Each step of covering in this process (namely, that of covering the function described in terms of order i implicants using order i+1 implicants) roughly corresponds to what Peng and Reggia call *layers* [20]. With the hope that work in this direction might identify closer ties between the two formalisms and also that theoretical results in each area translate into equivalent results in the other, we explored these ideas in greater detail. This subsection presents that work.

First, we make an important observation. Viewed in terms of abduction, the BMP turns out to be an independent problem. A product term such as $x \wedge \neg y$ covers $x \wedge \neg y \wedge z$ and $x \wedge \neg y \wedge \neg z$. Under this notion, the BMP is an independent abduction problem because clearly the union (representing the disjunction) of multiple product terms covers the union of the terms covered independently by each product term. Since the union covers no more than the union of what is covered independently, the BMP does not belong to the category of monotonic problems that are not independent. It may also be seen that the BMP is not an incompatibility problem either, because no two product terms cancel each other's effect in the sense defined in Section 3. In light of these observations, the mapping of Table 1 is not surprising because parsimonious covering theory (henceforth PCT) sets out to model independent problems that are not incompatibility problems.

An important question that arises now is: do the transitivity results of layered parsimonious covering problems [20] also extend into the abductive formulation of the BMP? (Let r(X,Y) stand for the relation "X is the set of all irredundant covers of Y." Our notion of trasitivity is a little weaker than the standard one, in conformity with [20]: whenever r(S3,S2) and r(S2,S1), S3 includes (but may not equal) all irredundant covers of S1.) Peng and Reggia came up with three negative results and a positive one: they showed that isolated minimal covers and isolated irredundant covers *are not transitive* across layers, but when the complete set of irredundant covers of the observed manifestations are derived at each layer, they *are transitive* across layers. They also showed that complete sets of minimal covers are not transitive across layers.

In other words, what they showed was that *any* irredundant cover at layer i+2 of *every* irredundant cover at layer i+1 of some entities at layer i is *not necessarily* an irredundant cover of the entities. Still, *some* irredundant cover at layer i+2 of *some* irredundant cover at layer i+1 of some entities at layer i is *guaranteed to be* an irredundant cover of the entities. In the case of the BMP as well, similar properties can be shown.

Theorem 3 *Isolated minimal covers and isolated irredundant covers are not transitive. Complete sets of minimal covers are not transitive, either.*

Proof: By counterexample below.

Example 3 This example refers to Figure 2. In the figure, the minterms constituting the function to be minimized are shown at layer 1. These minterms contain four literals in them and may be viewed as order 0 implicants for this particular problem. Layer 2 shows all relevant order 1 implicants, i.e., product terms with three literals in them that are implicants of the Boolean function in question. (They may be obtained by combining pairs of the previous order product terms that are combinable, i.e., those that are identical except for one variable that occurs complemented in one and

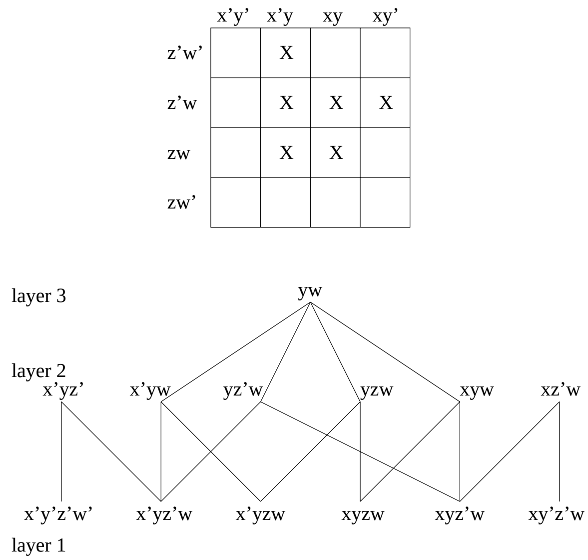

Figure 2: Irredundant (minimal) covers are not transitive

uncomplemented in the other.) At layer 2, it may be seen that there is an irredundant cover (which is also minimal), namely, $x'yz' + xz'w + yzw$. There is also a non-minimal irredundant cover at layer 2, namely, $x'yz' + xz'w + x'yw + xyw$. Now, the only implicant of the original Boolean function at layer 3 is yw, which cannot be obtained by combining any pair of terms in the minimal cover at layer 2 (since no pair of those terms is combinable). However, this layer 3 implicant *can* be obtained by combining $x'yw$ and xyw in the non-minimal irredundant cover of layer 2. Since $x'yz' + xz'w + yw$ is the only irredundant and minimal cover that includes any layer 3 terms, we conclude that isolated minimal covers are *not* transitive in the BMP case, as well. Since, every minimal cover is, as we have said, an irredundant cover, the example in discussion also serves as a counter-example for the other negative result: that isolated irredundant covers are *not* transitive, either. The negative result for complete sets of minimal covers also follows from this counterexample in a trivial way. •

Theorem 4 *Complete sets of irredundant covers are transitive across layers.*

Proof: To show that the complete sets of irredundant covers are transitive, it is sufficient to show that *every* irredundant cover at layer i+1 can be obtained from *some* irredundant cover at layer i. Suppose this were not true. There are two cases to consider:

Case 1: Some layer i+1 irredundant cover does not irredundantly cover any layer i irredundant cover. This means that the layer i+1 cover is a redundant cover of every layer i irredundant cover that it can cover. If this were the case, it could not have been an irredundant cover at layer i+1 in the first place! Thus we arrive at a contradiction.

Case 2: There is no irredundant cover at layer i which is irredundantly covered by some layer i+1 cover. This means that every layer i cover covered by the layer i+1 irredundant cover is redundant. Take one such layer i redundant cover. Obtain an irredundant cover by taking a suitable subset from it. The original layer i+1 irredundant cover either (a) still irredundantly covers this irredundant cover or (b) no longer *irredundantly* covers it. In subcase (a), we have a direct contradiction to the first sentence of this paragraph. In subcase (b), too, we have a contradiction in that the layer i+1 cover we considered could not, after all, have been irredundant. •

Nonmonotonic Aspect of Boolean Minimization

Abductive inference is known to be nonmonotonic in nature. In other words, some or all elements of the best explanation for a certain set of observations may no longer be part of the best explanation when additional observations are made. This is reflected in some of the tested cases of Boolean minimization. Recalling that explanation elements correspond to prime implicants in Boolean minimization, we think of the minimization process as nonmonotonic in the following sense: On adding one or more minterms to the original function, the new minimal form, viewed as a set of prime implicants, turns out to be *not a superset* of the minimal form of the original function. Furthermore, as a consequence of nonmonotonicity, it does not seem possible to compute the minimal form incrementally.

The function $x'yz'w' + x'yz'w + x'yzw + xyzw + xyz'w + xy'z'w$ is minimized as $x'yz' + yw + xz'w$. However, when an additional minterm $x'y'zw$ is included, the minimal form monotonically increased to $x'yz' + yw + xz'w + x'zw$. However, when yet another term $xyzw'$ is added, as already discussed, the minimal form turned out to be $x'yz' + xyw + xz'w + x'zw$, displaying a nonmonotonic nature. Yet, once again, when another minterm $xy'zw'$ is added, the function is minimized as $x'yz' + yw + xz'w + x'zw + xzw'$, reverting to a form very similar to one of the earlier minimal forms.

6 Related Work

We now sketch how our work on relating abduction to Boolean Minimization compares to that of others. In Reiter's work on modeling diagnosis from first principles [24], informally, a "symptom" corresponds to a "logical inconsistency", the "disjunction of disorders associated with each symptom" to a "conflict set", and a (minimal) "cover" to a (minimal) "hitting set". (Refer to [24] [9] for formal definitions.) In this new light, our algorithm generates all "covers" (and hence, all "minimal covers") by multiplying out the "disjunctions of disorders associated with symptoms", while Reiter's algorithm generates all hitting sets incrementally. In the latter approach, the "multiplication of disjunctions" is carried out in such a way that computing the first hitting set is quick and easy, but there is no guarantee that it will be minimal.

The results in [9] prove that a kernel (resp. minimal) diagnosis corresponds to a prime implicant of the set of minimal (resp. positive minimal) conflicts. In fact, in the spirit of [7], we showed that the set of minimal covers (that is, diagnoses) for a Parsimonious Covering Problem (henceforth, PCP) is precisely the *unique* set of prime implicants for the corresponding encoding of PCP as a BMP. A consequence of this result is that future improvements in boolean minimization algorithms can have direct impact on efficient computation of minimal covers for PCP. On the other hand, we are not aware of any simple *equivalent* encoding of BMP as a PCP.

The works of [19, 21, 22, 23] is concerned with abductive reasoning in the logic programming framework. In particular, [19] uses Horn-clauses (rather than propositional formulae [20] [9]) to encode domain information, and an ATMS (Assumption-Based Truth Maintenace System) to cache partial results, for efficient explanation generation. Their "heuristic" algorithm is incomplete/approximate. The PCP dicussed here generates clauses of the restricted form: "symptom ← disorder". For these cases, we do not expect to benefit much from the results of [19].

Recall that, for every propositional Horn clause set there is an equivalent subset that is devoid of any (mutual) recursion. We now relate our work to different formalizations of abduction available in the context of propositional Horn Clauses [21] [28].

Consider the following example from Chemistry: A base transforms a red litmus paper blue, while an acid transforms a blue litmus red. Both NaOH and KOH are bases.

$litmus\_test(X, red\_to\_blue) \leftarrow base(X)$
$litmus\_test(X, blue\_to\_red) \leftarrow acid(X)$
$base(X) \leftarrow KOH(X)$
$base(X) \leftarrow NaOH(X)$

Given that $litmus\_test(s\_id, red\_to\_blue)$ holds, the hypothesis that $NaOH(s\_id)$ holds can explain the observation. A predicate, whose literals can appear in an explanation, is called *abducible*.

1. *Most Specific Abduction*: This scheme was proposed in the context of medical diagnosis where the abducible predicates are the "basic causes" that appear only in the body of a rule, and not in any rule head [20]. According to this scheme, $\{KOH(s\_id)\}$ and $\{NaOH(s\_id)\}$ are the only valid explanations for the observation $\{litmus\_test(s\_id, red\_to\_blue)\}$.

2. *Predicate Specific Abduction*: In a number of applications such as circuit fault diagnosis [24], temporal reasoning [27] etc, it is possible to designate a distinguished set of predicates as abducibles. In our example all the predicates that stand for classes of compounds, such as *base*, KOH, $NaOH$, etc, can be treated as abducibles. Then, the observation $\{litmus\_test(s\_id, red\_to\_blue)\}$ can be explained by hypothesizing any one of the following: $\{base(s\_id)\}$, $\{KOH(s\_id)\}$ and $\{NaOH(s\_id)\}$

An instance of *most specific abduction problem* can be reduced to BMP containing only *abducible predicates*, by rewriting observations using the clauses *a la Prolog*. Furthermore, the *predicate specific abduction* [24] [27] can be viewed as an approach that makes explicit application-specific intuitions used in deriving a "best" explanation from a set of explanations.

Two-level logic minimization of a boolean expression seeks a logic representation with a minimum number of implicants and literals. According to [4], the generation of all prime implicants is exponential in the number of input variables and the extraction of minimum cover is polynomial in the number of prime implicants. A number of heuristics implemented in Espresso, a tool for boolean minimization, have been presented in [4].

As shown in [11], major variants of logic-based abduction, including propositional theories and even Horn Theories with subset-minimal explanations, minimum cardinality explanations, etc, are all computationally hard and there is no hope for complete and efficient algorithms to solve these problems. Similarly, [2, 5] proves that except for finding an explanation for independent or monotonic abduction problem, which is polynomial, almost all other variants such as for finding best explanations or all explanations or for incompatibility abduction problems, the problem is NP-hard [12]. In fact, [11] even suggest that, in the face of these negative results, it makes sense to tradeoff expressiveness and/or completeness and/or minimality requirement for tractability, to obtain approximate algorithms.

7 Conclusions

In this paper, we explicated the relationship between abduction and Boolean minimization. In particular, the BMP was mapped onto parsimonious covering theory that models only the simplest category of abduction problems, namely, independent problems. Since the BMP also turns out to be in the same category of independent problems, the match is perfect! One of the important strengths of parsimonious covering is its ability to model chained knowledge structures (e.g., causal chaining), and once again the layered nature of the BMP where implicants of increasing orders may be viewed as forming successive layers of minimization fits very well into chained knowledge structures. These close correspondences allow us to translate interesting properties from the layered parsimonious covering framework into Boolean minimization, and also to develop a corresponding minimization algorithm.

It may be noted that *only* the prime implicants of a given Boolean function in a BMP, rather than any general product terms, are considered analogous to disorders in an AP. The specific choice of prime implicants as the BMP-analog of disorders captures the important notion of logical equivalence of the minimal cover to the original Boolean expression. BMP consists of two major steps. In Section 4, an AP has been mapped into the first step of a BMP (the determination of all prime implicants). The second step of a BMP (the selection of a minimal or irredundant subset of prime implicants that can cover the given Boolean function) that mapped into an AP suggests the possibility that the complete BMP may be equivalent to the AP.

Our formulation of BMP in terms of PCT differs from Peng and Reggia's theory in two ways: in the way covering is defined to capture the notion of logical equivalence, and in the way layers are defined. An important consequence of this seems to be that, as covering at higher layers is performed, covers automatically get smaller.

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