An Algorithm for Computing Inconsistency Measurement by Paraconsistent Semantics *

Yue Ma[1,2], Guilin Qi[2], Pascal Hitzler[2], and Zuoquan Lin[1]

[1] Department of Information Science, Peking University, China
[2] Institute AIFB, Universität Karlsruhe, Germany
{mayue,lz}@is.pku.edu.cn, {yum,gqi,hitzler}@aifb.uni-karlsruhe.de

Abstract. Measuring inconsistency in knowledge bases has been recognized as an important problem in many research areas. Most of approaches proposed for measuring inconsistency are based on paraconsistent semantics. However, very few of them provide an algorithm for implementation. In this paper, we first give a four-valued semantics for first-order logic and then propose an approach for measuring the degree of inconsistency based on this four-valued semantics. After that, we propose an algorithm to compute the inconsistency degree by introducing a new semantics for first order logic, which is called $S[n]$-4 semantics.

1 Introduction

Measuring inconsistency in knowledge bases has been recognized as an important problem in many research areas, such as artificial intelligence [1,2,3,4,5], software engineering [6] and the Semantic Web [7]. There mainly exist two classes of inconsistency measures. The first class is defined by the number of formulas which are responsible for an inconsistency [8]. The second class considers propositions in the language which are affected by inconsistency [9,10,11,3,2]. The approaches belonging to the second class are often based on some paraconsistent semantics because we can still find paraconsistent models for inconsistent knowledge bases. The inconsistency degree considered in this paper belongs to the second class.

In [9], three compatible kinds of classifications for inconsistent theories are proposed, which actually provides three ways to define inconsistency measures for first-order logic based on paraconsistent semantics. The first approach is defined by the number of paraconsistent models. The underlying idea is that the less models, the more inconsistent the knowledge base is. The second approach is defined by the number of contradictions in a *preferred* paraconsistent model which has least contradictions, and considering the number of non-contradictions in a preferred model which has most non-contradictions. The third approach is defined by the number of atomic formulae which have conflicting assignments and by the number of all ground atomic formulae. Among

* We acknowledge support by the China Scholarship Council(http://www.csc.edu.cn/), by the German Federal Ministry of Education and Research (BMBF) under the SmartWeb project (grant 01 IMD01 B), by the EU under the IST project NeOn (IST-2006-027595, http://www.neon-project.org/), and by the Deutsche Forschungsgemeinschaft (DFG) in the ReaSem project.

K. Mellouli (Ed.): ECSQARU 2007, LNAI 4724, pp. 91–102, 2007.

these three approaches, the first one is global in the sense that all paraconsistent models are considered, while the latter two are local since they only consider the models with least inconsistencies or most consistencies. Later on, an approach for measuring inconsistency in first-order logic is given in [3], which is based on the third approach in [9].

Although there exist many approaches to measuring inconsistency in a knowledge base in a logical framework, very few of them provide efficient algorithms for implementation. In this paper, we first give a four-valued semantics for first-order logic and then propose an approach for measuring the degree of inconsistency based on this four-valued semantics. Our definition of inconsistency degree is similar to the approach given in [3]. The difference is that our approach is based on four-valued semantics and their approach is based on first-order quasi-classical semantics. After that, we propose an algorithm to compute the inconsistency degree by introducing a new semantics for first order logic, which is called $S[n]$-4 semantics.

This paper is organized as follows. In the next section, we introduce the four-valued semantics of first-order logic and its properties. In Section 3, we propose a definition of inconsistency degree of a first-order theory, and then, in Section 4, we give an algorithm to compute the inconsistency degree. Finally, we conclude the paper and discuss future work in Section 5.

2 Four-Valued First-Order Models

In order to measure the inconsistency degree of a first-order theory, in this section we define four-valued models for first-order theories. The inconsistency measurement studied in [2] is also by four-valued models. However, quantifiers and variables are not considered there — that is, only four-valued propositional models are being used. For first-order theories, an alternative semantic structure studied in [9] can be viewed as a three-valued semantics. Besides the definition of four-valued models, we also study how to reduce four-valued entailment to classical first-order entailment in this section, which serves as one of the important bases for our algorithm.

Given a set of predicate symbols $\mathcal{P}$ and a set of function symbols $\mathcal{F}$ (the set of 0-ary functions is a set of constant symbols, denoted $\mathcal{C}$), formulas are built up in the same way as in classical first-order logic from predicates, functions, a set of variables $\mathcal{V}$ and the set of logical symbols $\{\neg, \vee, \wedge, \forall, \exists, \rightarrow, \equiv\}$, where $\alpha \rightarrow \beta$ is the short form of $\neg\alpha \vee \beta$.

A first-order theory considered in this paper is a finite set of first-order formulae without free variables. In this paper, whenever we want to clarify the arity of a function or predicate, we may state the arity in parentheses following the function or predicate symbol, e.g. $f(n), P(n)$ means f, P are n-ary function and predicate, respectively. We also use t (possibly with subscripts) for terms, Greek lowercase symbols α, ϕ for formulas, and uppercase Γ for a first-order theory. The set of all predicates occurring in Γ is denoted as $\mathcal{P}(\Gamma)$. The cardinality of a set A is denoted by $|A|$.

The set of truth values for four-valued semantics [12,13] contains four elements: *true, false, unknown* (or *undefined*) and *both* (or *overdefined, contradictory*). We use the symbols t, f, N, B, respectively, for these truth values. The four truth values together with the ordering $\preceq$ defined below form a lattice $\mathsf{FOUR} = (\{t, f, B, N\}, \preceq)$:

$$f \preceq N \preceq t, f \preceq B \preceq t, N \text{ and } B \text{ are incomparable.}$$

The upper and lower bounds of two elements based on the ordering, and the operator $\neg$ on the lattice, are defined as follows:

- $N \wedge t = N, B \wedge t = B, N \wedge B = f$, and for any $x \in$ FOUR, $f \wedge x = f$;
- $f \vee N = N, f \vee B = B, N \vee B = t$, and for any $x \in$ FOUR, $t \vee x = t$;
- $\neg t = f, \neg f = t, \neg N = N, \neg B = B$, and for all $x \in$ FOUR, $\neg\neg x = x$.

Formally, a four-valued interpretation $\mathfrak{I}$ of a first-order theory is defined as follows.

Definition 1. *A four-valued interpretation $\mathfrak{I} = (\Delta^{\mathfrak{I}}, \cdot^{\mathfrak{I}})$ contains a non-empty domain $\Delta^{\mathfrak{I}}$ and a mapping $\cdot^{\mathfrak{I}}$ which assigns*

- *to each constant c an element of $\Delta^{\mathfrak{I}}$, written $c^{\mathfrak{I}}$;*
- *to each truth value symbol the symbol itself: $t^{\mathfrak{I}} = t, f^{\mathfrak{I}} = f, B^{\mathfrak{I}} = B, N^{\mathfrak{I}} = N$.*
- *to each n-ary function symbol $f(n)$ an n-ary function on $\Delta^{\mathfrak{I}}$, written $f^{\mathfrak{I}} : (\Delta^{\mathfrak{I}})^n \mapsto \Delta^{\mathfrak{I}}$, where $(\Delta^{\mathfrak{I}})^n = \overbrace{\Delta^{\mathfrak{I}} \times ... \times \Delta^{\mathfrak{I}}}^{n}$*
- *to each n-ary predication symbol $P(n)$ a pair of n-ary relations on $\Delta^{\mathfrak{I}}$, written $\langle P_+, P_- \rangle$, where $P_+, P_- \subseteq (\Delta^{\mathfrak{I}})^n$.*

Recall that a classical first-order interpretation maps each n-ary predicate to an n-ary relation on the domain. A four-valued interpretation assigns a pairwise n-ary relation $\langle P_+, P_- \rangle$ to each n-ary predicate P, where P_+ explicitly denotes the set of n-ary vectors which have the relation P under interpretation $\mathfrak{I}$ and P_- explicitly denotes the set of n-ary vectors which do not have the relation P under interpretation $\mathfrak{I}$. If a four-valued interpretation $\mathfrak{I}$ satisfies $P_+ \cup P_- = \Delta^{\mathfrak{I}}$ and $P_+ \cap P_- = \emptyset$, then it is a classical interpretation.

The definition of a state σ remains the same as in classical semantics of first-order logic, which is a mapping assigning to each variable occurring in $\mathcal{V}$ an element of the domain. Due to space limitation, we omit its formal definition as well as the definition of interpretation of terms based on states. We denote by $\sigma\{x \mapsto d\}$ the state obtained from σ by assigning d to x while leaving other assignments to other variables unchanged.

Given an interpretation $\mathfrak{I}$ and a state σ, the four-valued semantics of an atomic formula can be defined as follows.

Definition 2. *Assume $P(t_1, ..., t_n)$ is an n-ary predicate, where $t_1, ..., t_n$ are terms. $\mathfrak{I}$ is a four-valued interpretation and σ is a state. Then the truth value assignment to atomic predicates and equality is defined as follows:*

$$
\begin{aligned}
(x \equiv y)^{\mathfrak{I},\sigma} = t, &\ \textit{if and only if}\ \ x^\sigma = y^\sigma \\
(x \equiv y)^{\mathfrak{I},\sigma} = f, &\ \textit{if and only if}\ \ x^\sigma \neq y^\sigma \\
(P(t_1, ..., t_n))^{\mathfrak{I},\sigma} = t, &\ \textit{if and only if}\ \ (t_1^\sigma, ..., t_n^\sigma) \in P_+^{\mathfrak{I}}\ \textit{and}\ (t_1^\sigma, ..., t_n^\sigma) \notin P_-^{\mathfrak{I}} \\
(P(t_1, ..., t_n))^{\mathfrak{I},\sigma} = f, &\ \textit{if and only if}\ \ (t_1^\sigma, ..., t_n^\sigma) \notin P_+^{\mathfrak{I}}\ \textit{and}\ (t_1^\sigma, ..., t_n^\sigma) \in P_-^{\mathfrak{I}} \\
(P(t_1, ..., t_n))^{\mathfrak{I},\sigma} = B, &\ \textit{if and only if}\ \ (t_1^\sigma, ..., t_n^\sigma) \in P_+^{\mathfrak{I}}\ \textit{and}\ (t_1^\sigma, ..., t_n^\sigma) \in P_-^{\mathfrak{I}} \\
(P(t_1, ..., t_n))^{\mathfrak{I},\sigma} = N, &\ \textit{if and only if}\ \ (t_1^\sigma, ..., t_n^\sigma) \notin P_+^{\mathfrak{I}}\ \textit{and}\ (t_1^\sigma, ..., t_n^\sigma) \notin P_-^{\mathfrak{I}}
\end{aligned}
$$

where $\equiv$ is used for equality in first-order logic.

Note that the truth assignment to equality is classical in the sense that an equality can only obtain classical truth values t or f, while for common atomic predicates it may be valued among $\{t, f, B, N\}$. Based on the semantics of atomic predicates, the semantics of complex formulae can be defined deductively as follows:

Definition 3. *Let φ and ϕ be two first-order formulae, $\gamma(x_1, ..., x_n)$ be a formula containing n free variables, $\mathfrak{I}$ is a four-valued interpretation and σ is a state. Then,*

$$(\neg\varphi)^{\mathfrak{I},\sigma} = \neg(\varphi)^{\mathfrak{I},\sigma}; (\varphi \wedge \phi)^{\mathfrak{I},\sigma} = \varphi^{\mathfrak{I},\sigma} \wedge \phi^{\mathfrak{I},\sigma}; (\varphi \vee \phi)^{\mathfrak{I},\sigma} = \varphi^{\mathfrak{I},\sigma} \vee \phi^{\mathfrak{I},\sigma}$$

$$(\forall x_1, ..., x_n.\gamma(x_1, ..., x_n))^{\mathfrak{I},\sigma} = \bigwedge_{\sigma'=\sigma\{x_1 \mapsto d_1, ..., x_n \mapsto d_n\}} (\gamma(d_1, ..., d_n))^{\mathfrak{I},\sigma'}$$

$$(\exists x_1, ..., x_n.\gamma(x_1, ..., x_n))^{\mathfrak{I},\sigma} = \bigvee_{\sigma'=\sigma\{x_1 \mapsto d_1, ..., x_n \mapsto d_n\}} (\gamma(d_1, ..., d_n))^{\mathfrak{I},\sigma'}$$

Throughout the paper, we use the finite domain assumption such that the righthand of the last two equations above are finite conjunctions and disjunctions, respectively.

A four-valued interpretation $\mathfrak{I}$ is a 4-model of a first-order theory Γ if and only if for each formula $\alpha \in \Gamma$, $\alpha^{\mathfrak{I}} \in \{t, B\}$. A theory which has a 4-model is called 4-valued satisfiable. Four-valued entailment for first-order logic can be defined in a standard way as follows.

Definition 4. *Suppose Γ is a first-order theory and α is a first-order formula. Γ 4-valued entails α, written $\Gamma \models_4 \alpha$, if and only if every 4-model of Γ is a 4-model of α.*

Note that the four-valued interpretation of equality is the same as in classical first-order logic. So for all positive integers n, a four-valued interpretation $\mathfrak{I} = (\Delta^{\mathfrak{I}}, \cdot^{\mathfrak{I}})$ is a 4-model of formula $E_n = \exists x_1, ..., x_n. \bigwedge_{1 \leq i,j \leq n}(x_i \not\equiv x_j) \wedge \forall y. \bigvee_{1 \leq i \leq n}(y \equiv x_i)$ if and only if $|\Delta^{\mathfrak{I}}| = n$.

Proposition 1. *Given a first-order theory Γ without equality $\equiv$ and without boolean constants $\{t, f\}$, Γ always has 4-models of any domain size if UNA (the unique name assumption[1]) is not considered. If UNA is used, Γ always has 4-models whose sizes are equivalent to or larger than the number of constants in Γ.*

Example 1. *(Canonical example) $\Gamma = \{Penguin(tweety), Bird(fred), \forall x.Bird(x) \rightarrow Fly(x), \forall x.Penguin(x) \rightarrow Bird(x), \forall x.Penguin(x) \rightarrow \neg Fly(x)\}$. Obviously, Γ has no two-valued models. However, it has the following 4-model $\mathfrak{I} = (\Delta^{\mathfrak{I}}, \cdot^{\mathfrak{I}})$, where $\Delta^{\mathfrak{I}} = \{a, b\}$ and $\cdot^{\mathfrak{I}}$ is defined as $tweety^{\mathfrak{I}} = a, fred^{\mathfrak{I}} = b, Fly^{\mathfrak{I}}(a) = B, Penguin^{\mathfrak{I}}(a) = Bird^{\mathfrak{I}}(a) = Bird^{\mathfrak{I}}(b) = Fly^{\mathfrak{I}}(b) = t, Penguin^{\mathfrak{I}}(b) = f$.*

According to Proposition 1, we restrict our measurement of the inconsistency degree to first-order theories which do not contain equality or $\{t, f\}$ in this paper.

Our four-valued semantics is an extension of classical semantics. Additionally, 4-valued entailment can be reduced to the classical entailment. The reduction in the propositional case is studied in [14]. We extend it to the first-order case.

[1] That is, if c and d are distinct constants, then $c^{\mathfrak{I}} \neq d^{\mathfrak{I}}$ for each interpretation $\mathfrak{I}$.

Theorem 1. *Let Γ be a first-order theory in negation normal form and ϕ be a formula. $\Gamma \models_4 \phi$ if and only if $\Theta(\Gamma) \vdash \Theta(\phi)$, where $\Theta(\cdot)$ is a function defined on a set of formulae as follows:*

- $\Theta(c) = c$, *if c is a constant.*
- $\Theta(\varphi) = \varphi$, *if φ is $x \equiv y$ or $x \not\equiv y$;*
- $\Theta(P(x_1, ..., x_n)) = P^+(x_1, ..., x_n)$, *where P^+ is a new atomic n-ary predicate;*
- $\Theta(\neg P(x_1, ..., x_n)) = P^-(x_1, ..., x_n)$, *where P^- is a new n-ary predicate;*
- $\Theta(\varphi_1(x_1, ..., x_n) \circ \varphi_2(y_1, ..., y_m)) = \Theta(\varphi_1(x_1, ..., x_n)) \circ \Theta(\varphi_2(y_1, ..., y_m))$, *where $\circ$ is $\wedge$ or $\vee$;*
- $\Theta(\varphi_1(x_1, ..., x_n) \to \varphi_2(y_1, ..., y_m)) = \Theta(\neg\varphi_1(x_1, ..., x_n)) \vee \Theta(\varphi_2(y_1, ..., y_m))$.
- $\Theta(Qx.\varphi) = Qx.\Theta(\varphi)$, *where Q is $\forall$ or $\exists$.*
- $\Theta(\Gamma) = \{\Theta(\varphi) \mid \varphi \in \Gamma\}$.

Example 2. *(Example 1 continued)*
$\Theta(\Gamma) = \{\mathit{Penguin}^+(tweety), \mathit{Bird}^+(freg), \forall x.\mathit{Bird}^-(x) \vee \mathit{Fly}^+(x), \forall x.\mathit{Penguin}^-(x) \vee \mathit{Bird}^+(x), \forall x.\mathit{Penguin}^-(x) \vee \mathit{Fly}^-(x)\}$.

Example 3. *(Example 2 continued)*
Consider $\Gamma' = \Gamma \wedge \mathit{Fly}(a_1) \wedge \neg \mathit{Fly}(a_1) \wedge E_n$ *and* $\varphi = \bigvee_{2 \le j \le n}(\mathit{Fly}(a_j) \wedge \neg \mathit{Fly}(a_j)) \vee \bigvee_{1 \le j \le n}((\mathit{Bird}(a_j) \wedge \neg \mathit{Bird}(a_j)) \vee (\mathit{Penguin}(a_j) \wedge \neg \mathit{Penguin}(a_j)))$. *Obviously,* $\Theta(\Gamma') = \Theta(\Gamma) \wedge \mathit{Fly}^+(a_1) \wedge \mathit{Fly}^-(a_1)) \wedge E_n$ *and* $\Theta(\varphi) = \bigvee_{2 \le j \le n}(\mathit{Fly}^+(a_j) \wedge \mathit{Fly}^-(a_j)) \vee \bigvee_{1 \le j \le n}((\mathit{Bird}(a_j)^+ \wedge \mathit{Bird}^-(a_j)) \vee (\mathit{Penguin}^+(a_j) \wedge \mathit{Penguin}^-(a_j)))$. *According to Theorem 1, we know that* $\Gamma' \not\models_4 \varphi$ *because* $\Theta(\Gamma') \not\vdash \Theta(\varphi)$. *This example will be again used in Example 7.*

3 Inconsistency Measure by 4-Valued Semantics

To measure inconsistency of a theory, we consider only finite theory and only finite domains in this paper. This is reasonable for practical cases because only finite individuals can be represented or would be used.

Our approach to measuring inconsistency is based on the approach given in [3] which is defined by means of first-order quasi-classical models instead of four-valued models. The reason why we use 4-valued models is that the 4-valued semantics for the whole first-order language can be implemented by a linear reduction to the classical semantics. While for quasi-classical logic, this is only achieved restricted to propositional logic in CNF [15]. Due to space limitation, we omit all proofs. The underlying idea comes from [3].

Definition 5. *Let Γ be a first-order theory and $\mathfrak{I} = (\Delta^{\mathfrak{I}}, \cdot^{\mathfrak{I}})$ be a four-valued model of Γ. The* inconsistency degree *of Γ w.r.t. $\mathfrak{I}$, denoted $Inc_{\mathfrak{I}}(\Gamma)$, is a value in $[0, 1]$ calculated in the following way:*

$$Inc_{\mathfrak{I}}(\Gamma) = \frac{|\mathit{ConflictTheo}(\mathfrak{I}, \Gamma)|}{|\mathit{GroundTheo}(\mathfrak{I}, \Gamma)|}$$

where $\mathit{GroundTheo}(\mathfrak{I}, \Gamma) = \{P(d_1, ..., d_n) \mid d_1, ..., d_n \in \Delta^{\mathfrak{I}}, P(n) \in \mathcal{P}(\Gamma)\}$, *and* $\mathit{ConflictTheo}(\mathfrak{I}, \Gamma) = \{(P(d_1, ..., d_n))^{\mathfrak{I}} = B \mid d_1, ..., d_n \in \Delta^{\mathfrak{I}}, P(n) \in \mathcal{P}(\Gamma)\}$.

That is, the inconsistency degree of Γ w.r.t. $\Im$ is the ratio of the number of conflicting atomic sentences divided by the amount of all possible atomic sentences formed from atomic predicates occurring in Γ and individuals in the domain of $\Im$. It measures to what extent a given first-order theory Γ contains inconsistencies w.r.t. $\Im$.

Example 4. *(Example 1 continued) GroundTheo*$(\Im, \Gamma) = \{$*Bird*(a), *Penguin*(a), *Fly*(a), *Bird*(b), *Penguin*(b), *Fly*$(b)\}$, *ConflictTheo*$(\Im, \Gamma) = \{$*Fly*$(a)\}$. *So* $Inc_{\Im}(\Gamma) = \frac{1}{6}$.

Let's consider another 4-valued model $\Im'$ *of* Γ: $tweety^{\Im'} = a, fred^{\Im'} = b, Fly^{\Im'}(a) = Penguin^{\Im'}(a) = Bird^{\Im'}(a) = Bird^{\Im'}(b) = Fly^{\Im'}(b) = B, Penguin^{\Im'}(b) = f$. *Obviously,* *GroundTheo*$(\Im', \Gamma) =$ *GroundTheo*$(\Im, \Gamma)$, $|$*GroundTheo*$(\Im', \Gamma)| = 5$, *and* $Inc_{\Im'}(\Gamma) = \frac{5}{6}$.

From this example, we can see that for any given first-order theory, its different 4-valued models might contain different percents of contradictions. According to this, we define a partial ordering on the set of its models as follows.

Definition 6. *(Model ordering w.r.t. inconsistency) Let* $\Im_1$ *and* $\Im_2$ *be two four-valued models of a first-order theory* Γ *such that* $|\Delta_1^{\Im}| = |\Delta_2^{\Im}|$. *We say that* $\Im_1$ *is less inconsistent than* $\Im_2$, *written* $\Im_1 \leq_{Incons} \Im_2$, *if and only if* $Inc_{\Im_1}(\Gamma) \leq Inc_{\Im_2}(\Gamma)$.

As usual, $\Im_1 <_{Incons} \Im_2$ denotes $\Im_1 \leq_{Incons} \Im_2$ and $\Im_2 \not\leq_{Incons} \Im_1$, and $\Im_1 \equiv_{Incons} \Im_2$ denotes $\Im_1 \leq_{Incons} \Im_2$ and $I_2 \leq_{Incons} \Im_1$. $\Im_1 \leq_{Incons} \Im_2$ means that $\Im_1$ is more consistent than $\Im_2$. The models of size n which are minimal w.r.t $\leq_{Incons}$ are called preferred models and they are formally defined as follows.

Definition 7. *Let* Γ *be a first-order theory,* $\mathcal{M}_4(\Gamma)$ *be the set of 4-models of* Γ, *and* $n(n \geq 1)$ *be a given cardinality.* Preferred models of size n w.r.t. $\leq_{Incons}$, *written* $PreferModel_n(\Gamma)$, *are defined as follows:*

$$PreferModel_n(\Gamma) = \{\Im \mid |\Delta^{\Im}| = n; \forall \Im' \in \mathcal{M}_4(\Gamma), |\Delta^{\Im'}| = n \text{ implies } \Im \leq_{Incons} \Im'\}.$$

By Proposition 1 and Definition 7, it is not hard to see that given a first-order theory and an integer n, we can always find a preferred model if the unique name assumption is not used. Otherwise, with the unique name assumption, we only can find a preferred model provided n is not less than the number of constants appearing in the theory.

As a direct consequence of Definition 6 and Definition 7, the following corollary shows that for any two preferred four-valued models of a first-order theory with the same cardinality, the inconsistency degrees of the theory $w.r.t.$ them are equal.

Corollary 2. *Let* Γ *be a first-order theory and* $n(\geq 1)$ *be any given positive integer. Suppose* $\Im_1$ *and* $\Im_2$ *are two four-valued models of* Γ *such that* $|\Delta^{\Im_1}| = |\Delta^{\Im_2}| = n$, *and* $\{\Im_1, \Im_2\} \subseteq PreferModel_n(\Gamma)$. *Then* $Inc_{\Im_1}(\Gamma) = Inc_{\Im_2}(\Gamma)$.

Based on Corollary 2, the following definition of inconsistency degree of a first-order theory is well-defined.

Definition 8. *Given a first-order theory* Γ *and an arbitrary cardinality* $n(n \geq 1)$, *let* $\Im_n$ *be an arbitrary model in* $PreferModels_n(\Gamma)$. *The inconsistency degree of* Γ, *denoted by TheoInc*(Γ), *is defined as* $\langle r_1, r_2, ..., r_n, ...\rangle$, *where* $r_n = *$ *if* $PreferModel_n(\Gamma) = \emptyset$, *and* $r_n = Inc_{\Im_n}(\Gamma)$ *otherwise. We use* $*$ *as a kind of null value.*

Following [3], we also use a sequence as the inconsistency degree of a first-order theory. This sequence can reflect the inconsistency information of the theory with respect to each finite size domain. For such sequences, the following property holds obviously.

Proposition 2. *Given an inconsistent first-order theory Γ, assume $|\mathcal{C}|$ is the number of constants of Γ and TheoInc$(\Gamma) = \langle r_1, r_2, ...\rangle$. Then for $i \geq |\mathcal{C}|$, $r_i \neq *$ and $r_i > 0$.*

This proposition shows that for any given first-order theory, its inconsistency measure cannot be a meaningless sequence (i.e., each element is the null value $*$) no matter whether UNA is used or not. Moreover, the non-zero values in the sequence start at least from the position which equals the number of constants in the first-order theory, and remains greater than zero in the latter positions of the sequence.

Example 5. *(Example 1 continued) If UNA is used, TheoInc$(\Gamma) = \langle *, \frac{1}{6}, ..., \frac{1}{3n}, ...\rangle$. If UNA is not used, TheoInc$(\Gamma) = \langle \frac{1}{3}, \frac{1}{6}, ..., \frac{1}{3n}, ...\rangle$. The 4-models which only assign Fly$(tweety)$ to B are among the preferred models in both cases.*

4 Computational Aspects of Inconsistency Degree Sequences

A naive way to compute the inconsistency degree is to list all models to check which are the preferred models, and then compute the number of contradictions in these models. For a first-order theory, listing all models is not an easy and practical reasoning task.

In this section, we propose a practical way to compute the inconsistency degree by reducing the computation of the inconsistency degree to classical entailment, such that existing reasoners for first-order logic can be reused.

4.1 $S[n]$-4 Semantics

In this subsection, we define $S[n]$-4 semantics for first-order logic and show that $S[n]$-4 entailment can be reduced to classical entailment via four-valued entailment. We were inspired by [16]. $S[n]$-4 semantics will serve as the basis for our algorithm for computing the inconsistency degrees in Section 4.2.

Throughout this section, we assume that there is an underlying finite set of predicates $\mathcal{P}$ used for building all formulae and that $\mathcal{D}_n = \{a_1, ..., a_n\}$. The set of ground atomic formulae *Base*$(\mathcal{P}, \mathcal{D}_n)$ is defined as the set $\{P(a_{i_1}, ..., a_{i_m}) \mid P(m) \in \mathcal{P}, a_{i_1}, ..., a_{i_m} \in \mathcal{D}_n\}$.

Definition 9. *($S[n]$-4 Interpretation) Let $\mathcal{D}_n = \{a_1, ..., a_n\}$ be a domain of size n and S be any given subset of Base$(\mathcal{P}, \mathcal{D}_n)$. A 4-valued interpretation $\mathfrak{I}$ with domain $\mathcal{D}_n$ is called an $S[n]$-4 interpretation if and only if it satisfies the following condition:*

$$\phi^{\mathfrak{I}} = \begin{cases} B & \text{if } \phi \in \mathit{Base}(\mathcal{P}, \mathcal{D}_n) \setminus S, \\ N \text{ or } t \text{ or } f & \text{if } \phi \in S \text{ and } \{N, t, f\} \subseteq \mathit{FOUR} \end{cases}$$

That is, $\mathfrak{I}$ is an $S[n]$-4 interpretation if and only if it is a 4-valued interpretation with domain of size n and assigns the contradictory truth value B to the ground atomic formulae not in S, and it maps non-contradictory truth values to ground atomic formulae in S.

Definition 10. *Let Γ be a first-order theory. An $S[n]$-4 interpretation $\mathfrak{I}$ is an $S[n]$-4 model of Γ if and only if it is a 4-model of Γ. A theory is $S[n]$-4 satisfiable if and only if it has an $S[n]$-4 model.*

Example 6. *Let $\mathcal{P} = \{p(x), q(x, y)\}$, $n = 2$, $\mathcal{D}_2 = \{a_1, a_2\}$. Then Base$(\mathcal{P}, \mathcal{D}_2) = \{p(a_1), p(a_2), q(a_1, a_1), q(a_2, a_2), q(a_1, a_2), q(a_2, a_1)\}$. Consider $\Gamma = \{\exists x.(p(x) \wedge \neg p(x)), \forall x \exists y.q(x, y)\}$.*

- ◇ *Let $S_1 = \{p(a_2), q(a_1, a_1), q(a_2, a_2), q(a_1, a_2), q(a_2, a_1)\}$. Γ is $S_1[2]$-4 satisfiable and has the following $S_1[2]$-4 model $\mathfrak{I}$: $p^{\mathfrak{I}}(a_1) = B$, and $\varphi^{\mathfrak{I}} = t$ for all $\varphi \in S_1$.*
- ◇ *Let $S_2 = \{p(a_1), p(a_2)\}$. Γ is $S_2[2]$-4 unsatisfiable since all $S_2[2]$-4 interpretations should map neither $p(a_1)$ nor $p(a_2)$ to B, so $\exists x.p(x) \wedge \neg p(x)$ cannot be satisfied.*

Theorem 3. *(Monotonicity) For any positive integer n, assume the two sets S and S' satisfying $S \subseteq S' \subseteq$ Base$(\mathcal{P}, \mathcal{D}_n)$. If a theory Γ is $S[n]$-4 unsatisfiable, then it is $S'[n]$-4 unsatisfiable.*

Proof. Assume that Γ is $S[n]$-4 unsatisfiable and that there exists an $S'[n]$-4 interpretation $\mathfrak{I}_{S'}$ satisfying Γ. We construct an $S[n]$-4 interpretation $\mathfrak{I}_S$ as follows.

$$\phi^{\mathfrak{I}_S} = \begin{cases} B & \text{if } \phi \in S' \setminus S, \\ \phi^{\mathfrak{I}_{S'}} & \text{otherwise.} \end{cases}$$

Obviously, $\mathfrak{I}_S$ is an $S[n]$-4 model of Γ, which is a contradiction. □

Definition 11. *($S[n]$-4 entailment) A sentence ϕ is $S[n]$-4 implied by a theory Γ, denoted $\Gamma \models^4_{S[n]} \phi$, if and only if every $S[n]$-4 model of Γ is an $S[n]$-4 model of ϕ.*

The relation between $S[n]$-4 satisfiability and $S[n]$-4 entailment is obvious.

Proposition 3. *Γ is $S[n]$-4 unsatisfiable if and only if $\Gamma \models^4_{S[n]} f$, where $f \in$ FOUR.*

The following theorem shows that $S[n]$-4 entailment can be reduced to 4-valued entailment in first-order logic.

Theorem 4. *For any $n \geq 1$ and $S \subseteq$ Base$(\mathcal{P}, \mathcal{D}_n)$, let $S = \{\alpha_1, ..., \alpha_m\}$ and $T =$ Base$(\mathcal{P}, \mathcal{D}_n) \setminus S = \{\beta_1, ..., \beta_k\}$, where $m + k = n$. Then the following claim holds:*

$$\Gamma \models^4_{S[n]} \varphi \text{ if and only if } \Gamma \wedge \bigwedge_{1 \leq i \leq k} (\beta_i \wedge \neg \beta_i) \wedge E_n \models_4 \varphi \vee \bigvee_{1 \leq j \leq m} (\alpha_j \wedge \neg \alpha_j),$$

where $E_n = \exists x_1, ..., x_n. \bigwedge_{1 \leq i,j \leq n} (x_i \not\equiv x_j) \wedge \forall y. \bigvee_{1 \leq i \leq n} (y \equiv x_i)$.

The right side of the claim is explained as follows: for each 4-model I of Γ, if I satisfies

1. it has an n-size domain (i.e., E_n is satisfied by I) and
2. it assigns truth value B to each element in *Base*$(\mathcal{P}, \mathcal{D}_n) \setminus S$ (i.e., the conjunction $\bigwedge_{1 \leq i \leq k} (\beta_i \wedge \neg \beta_i)$ is satisfied by I),

then I is not an $S[n]$-4 model of Γ if and only if it assigns B to at least one element in S (i.e., the disjunction $\bigvee_{1 \leq j \leq m} (\alpha_j \wedge \neg \alpha_j)$ is true under I). A formal proof is as follows.

Proof. Let $\Gamma' = \Gamma \wedge \bigwedge_{1\leq i\leq k}(\beta_i \wedge \neg\beta_i) \wedge E_n$ and let $\varphi' = \varphi \vee \bigvee_{1\leq j\leq m}(\alpha_j \wedge \neg\alpha_j)$.

($\Rightarrow$) For any 4-model M_4 of Γ', we show that M_4 satisfies φ'. First, from the assumption that M_4 satisfies Γ', we know $|\Delta^{M_4}| = n$ and $M_4(\beta_i) = B$ for $1 \leq i \leq k$. If there is j_0, $1 \leq j_0 \leq m$ such that $M_4(\alpha_{j_0}) = B$, then M_4 is a 4-valued model of φ'. Otherwise, if for each $1 \leq j \leq m$, $M_4(\alpha_j) \neq B$, then M_4 is an $S[n]$-4 model of Γ, so M_4 satisfies φ by hypothesis and therefore satisfies φ'.

($\Leftarrow$) For any $S[n]$-4 model M_S of Γ, we show that M_S satisfies φ. By definition of M_S, $|\Delta^{M_S}| = n$, $M_S(\beta_i) = B$ for $1 \leq i \leq k$, and $M_S(\alpha_j) \neq B$ for $1 \leq j \leq m$. So M_S is a 4-model of Γ' but does not satisfy $\bigvee_{1\leq j\leq m}(\alpha_j \wedge \neg\alpha_j)$. Then M_S satisfies φ by hypothesis and $\Gamma \models_{S\text{-}4} \varphi$. □

Corollary 5. *Let $S = \{\alpha_1, ..., \alpha_m\}$ and let $T = Base(\mathcal{P}, \mathcal{D}_n) \setminus S = \{\beta_1, ..., \beta_k\}$. Γ is $S[n]$-4 unsatisfiable if and only if*

$$\Theta(\Gamma \wedge \bigwedge_{1\leq i\leq k}(\beta_i \wedge \neg\beta_i)) \wedge E_n \vdash \bigvee_{1\leq j\leq m}\Theta((\alpha_j \wedge \neg\alpha_j)).$$

Proof. This corollary holds by replacing ϕ with f in Theorem 4 and then performing $\Theta(\cdot)$ according to Theorem 1 with the fact that $\Theta(E_n) = E_n$.

This theorem shows that $S[n]$-4 satisfiability can be reduced to classical entailment in first-order logic.

4.2 Algorithm for Computing the Inconsistency Degree

In this section, we first study how the inconsistency degree of an inconsistent theory Γ can be characterized by $S[n]$-4 satisfiability. Secondly, we give an algorithm to compute the inconsistency degree by invoking a classical reasoner.

Without loss of generality, throughout this section, we assume that the n-size ($n \geq 1$) domain of any 4-valued interpretation is $\mathcal{D}_n = \{a_1, ..., a_n\}$. Whenever we talk about $S[n]$-4 semantics used to compute the inconsistency degree of a first-order theory Γ, we always assume that the underlying finite set of predicates $\mathcal{P}$ is all the predicates occurring in Γ — that is, $\mathcal{P} = \mathcal{P}(\Gamma)$ and $Base(\mathcal{P}, \mathcal{D}_n) = GroundTheo(\mathcal{D}_n, \Gamma)$.

Theorem 6. *Let $TheoInc(\Gamma) = \langle r_1, ..., r_n, ...\rangle$. If $r_n \neq *$, the equation*

$$r_n = 1 - \frac{B_n}{|GroundTheo(\mathcal{D}_n, \Gamma)|} \tag{1}$$

holds, where $B_n = max\{|S| : S \subseteq GroundTheo(\mathcal{D}_n, \Gamma)$, so that Γ is $S[n]$-4 satisfiable$\}$.

Proof. Let $\Im_n$ be a preferred model and S be the set of atomic sentences all of which are not assigned the contradictory value B under $\Im_n$. Therefore, Γ is $S[n]$-4 satisfiable because $\Im_n$ is already an $S[n]$-4 model of Γ. For any subset $S' \subseteq GroundTheo(\mathcal{D}_n, \Gamma)$ such that $|S'| > |S|$, we claim that Γ is $S'[n]$-4 unsatisfiable. Otherwise suppose $\Im_{S'}$ is an $S'[n]$-4 model of Γ. Obviously, $\Im_{S'} <_{Incons} \Im_n$, since $|S'| > |S|$, contradicting the definition of $\Im_n$. Thus $B_n = |GroundTheo(\mathcal{D}_n, \Gamma)| - |ConflictTheo(\Im_n, \Gamma)|$. By Definition 5 and Definition 8, Equation 1 holds. □

Theorem 6 shows that the computation of r_n can be reduced to the problem of computing the maximal cardinality of S such that S is a subset of *GroundTheo*$(\mathcal{D}_n, \Gamma)$ and Γ is $S[n]$-4 satisfiable. We are now ready to give an algorithm to compute each element of the inconsistency degree sequence of a first-order theory Γ. The underlying idea is that we test $S[n]$-4 satisfiability for each subset S of *GroundTheo*$(\mathcal{D}_n, \Gamma)$ from size $|$*GroundTheo*$(\mathcal{D}_n, \Gamma)| - 1$ to 1. Whenever such subset has been found, the value of r_n is calculated by Equation 1 and the procedure ends.

Algorithm 1. Computing_Inconsistency_Degree(Γ, n)

Input: An inconsistent first-order theory Γ and a positive integer n
Output: r_n // *TheoInc*$(\Gamma) = \langle r_1, ..., r_n, ...\rangle$

1: $N \leftarrow$ the number of constants in Γ
2: **if** $n < N$ and UNA is used **then**
3: $r_n \leftarrow *$
4: **return** r_n
5: **end if**
6: $\mathcal{D}_n \leftarrow \{a_1, ..., a_n\}$,
7: $\Sigma \leftarrow$ *GroundTheo*$(\mathcal{D}_n, \Gamma)$ // see *GroundTheo*$(\mathcal{D}_n, \Gamma)$ in Definition 5
8: $r_n = 0$ // The initial value of r_n is set to 0
9: **for** $l \leftarrow |\Sigma| - 1$ to 1 **do**
10: $S \leftarrow PopSubset(\Sigma, l)$
// $PopSubset(\cdot, \cdot)$ is a procedure to return a subset of Σ with cardinality l. Once a subset is returned, it will not be selected again.
11: **while** $S \neq \emptyset$ **do**
12: **if** Γ is $S[n]$-4 satisfiable **then**
13: $r_n \leftarrow (1 - \frac{l}{|\Sigma|})$ **exit**
14: // $|S| = \max\{|S'| \mid S' \subseteq$ *GroundTheo*$(\mathcal{D}_n, \Gamma)$, Γ is $S'[n]$-4 satisfiable $\}$.
15: **else**
16: $S \leftarrow PopSubset(\Sigma, l)$
17: **end if**
18: **end while**
19: **if** $r_n \neq 0$ **then**
20: **exit** // The subset used to compute r_n has been found w.r.t. size l
21: **else**
22: $l \leftarrow l-1$ // We have to find a subset used to compute r_n w.r.t. a smaller cardinality.
23: **end if**
24: **end for**
25: **if** $l = 0$ **then**
26: $r_n = 1$
27: **end if**
28: **return** r_n

In Algorithm 1, if UNA is used and the input n is strictly less than the number of constants in Γ, then $r_n = *$ is returned (see line 2 to line 5). If it is not the case, the initialization process follows till line 8. From line 9 to line 27 we have the main steps of the algorithm to compute the inconsistency degree, where subsets of *GroundTheo*$(\mathcal{D}_n, \Gamma)$ are selected one by one according to a decreasing size ordering, so that whenever the first subset S satisfying the condition in line 12, the inconsistency degree r_n is computed and the whole procedure ends. This is indeed the case because such S satisfies $B_n = |S| = \max\{|S'| \mid S' \subseteq \mathit{GroundTheo}(\mathcal{D}_n, \Gamma), \Gamma \text{ is } S'[n]\text{-4 satisfiable}\}$, where B_n is defined as in Theorem 6. Since Γ is inconsistent, it is no necessity to test $l = |\Sigma|$ in line 9. Furthermore, if no proper subset S of $\mathit{GroundTheo}(\mathcal{D}_n, \Gamma)$ can satisfy the condition in line 12, then this means that all sentences in $\mathit{GroundTheo}(\mathcal{D}_n, \Gamma)$ should be assigned B by preferred models, thus $r_n = 1$. This shows the correctness of this algorithm as well.

For line 12, the condition of $S[n]$-4 satisfiability can be decided by classical entailment of first-order logic according to Corollary 5, such that each r_n in the inconsistency degree sequence can be computed by invoking a classical reasoner. We give an example to illustrate Algorithm 1.

Example 7. *(Example 5 continued) We take the case that UNA is used and* $n \geq 2$. $\mathit{GroundTheo}(\mathcal{D}_n, \Gamma) = \{\mathit{Bird}(a_i), \mathit{Fly}(a_i), \mathit{Penguin}(a_i) \mid a_i \in \mathcal{D}_n\}$ *so that* $|\Sigma| = 3n$. *For* $l = |\Sigma| - 1 = 3n - 1$, *assume that following subset of* Σ *is selected:* $S = \mathit{GroundTheo}(\mathcal{D}_n, \Gamma) \setminus \{\mathit{Fly}(a_1)\}$. *We have that* Γ *is* $S[n]$*-4 satisfiable because of the result studied in Example 3. Then* $r_n = 1 - \frac{l}{3n} = \frac{1}{3n}$, *which equals the general representation of the inconsistency degree of* Γ *in Example 5.*

The computation of the inconsistency degree sequence $\langle r_1, ..., r_n, ...\rangle$ of a first-order theory Γ can be achieved using Algorithm 1. However, a practical problem is that the infinite style definition of $\mathit{TheoInc}(\Gamma)$ makes us unable to get the exact value of $\mathit{TheoInc}(\Gamma)$ in finite time. We can however set a termination condition in order to guarantee that an answer will be obtained. Suppose time (resource) is used up, a possible way is to use the already obtained partial sequences $\langle r_1, ..., r_n\rangle$ as an approximating value of $\mathit{TheoInc}(\Gamma)$.

From Theorem 6 and Corollary 5, the computation of each element of an inconsistency degree sequence includes at most $2^{|\Sigma|}$ times invoking a classical entailment, where $|\Sigma| \leq Kn^M$ for any $n \geq 1$ provided that the maximal arity of predicates in Γ is M and the number of predicates in Γ is K. The worst case occurs when all subsets of Σ have to be searched.

As to an optimization of the algorithm, the direct way is to properly design a procedure $PopSubset(\cdot, \cdot)$ such that the correct S which makes Γ $S[n]$-4 satisfiable can be found within as few steps as possible.

5 Conclusions and Future Work

In this paper, we have studied the computational aspects of calculating the inconsistency degree of a first-order theory. Theoretically, we have shown the process of encoding the calculation of the inconsistency degree as a first-order unsatisfiability decision problem via the $S[n]$-4 semantics proposed in this paper.

The semi-decidability of first-order logic makes Algorithm 1 semi-computes the inconsistency degree of first-order theory in the sense that we can be informed in finite time when Γ is $S[n]$-4 unsatisfiable for a chosen S; However if the correct subset of S such that Γ is $S[n]$-4 satisfiable is chosen, we actually cannot get the answer in finite time in general cases. Therefore we also have to set a time termination condition for each computation of r_n, and when time is used up, Γ can be roughly considered to be $S[n]$-4 satisfiable and we can use this S to compute r_n.

Considering the semi-decidability problem, the study of implementing our algorithm on Description Logics which include a family of decidable fragments of first-order logic becomes meaningful [17].

In the future, we will study how to extend the underlying idea of our algorithm to compute other approaches to measuring inconsistency, such as the inconsistency degree defined in [3]. In order to provide inconsistency degree information for real applications, we will also consider approximating approaches to measuring inconsistency in future work.

References

1. Hunter, A.: Reasoning with contradictory information using quasi-classical logic. J. Log. Comput. 10, 677–703 (2000)
2. Hunter, A.: How to act on inconsistent news: Ignore, resolve, or reject. Data Knowl. Eng. 57, 221–239 (2006)
3. Grant, J., Hunter, A.: Measuring inconsistency in knowledgebases. J. Intell. Inf. Syst. 27(2), 159–184 (2006)
4. Hunter, A., Konieczny, S.: Shapley inconsistency values. In: Proc. of KR'06, pp. 249–259 (2006)
5. Konieczny, S., Lang, J., Marquis, P.: Quantifying information and contradiction in propositional logic through epistemic tests. In: Proc. of IJCAI'03, pp. 106–111 (2003)
6. Mu, K., Jin, Z., Lu, R., Liu, W.: Measuring inconsistency in requirements specifications. In: Godo, L. (ed.) ECSQARU 2005. LNCS (LNAI), vol. 3571, pp. 440–451. Springer, Heidelberg (2005)
7. Deng, X., Haarslev, V., Shiri, N.: Measuring inconsistencies in ontologies. In: ESWC, pp. 326–340 (2007)
8. Knight, K.: Measuring inconsistency. Journal of Philosophical Logic 31, 77–98 (2001)
9. Grant, J.: Classifications for inconsistent theories. Notre Dame J. Formal Logic 19, 435–444 (1978)
10. Hunter, A.: Measuring inconsistency in knowledge via quasi-classical models. In: Proc. of AAAI/IAAI, pp. 68–73 (2002)
11. Oller, C.A.: Measuring coherence using LP-models. J. Applied Logic 2, 451–455 (2004)
12. Belnap, N.D.: A useful four-valued logic. Modern uses of multiple-valued logics, 7–73 (1977)
13. Arieli, O., Avron, A.: The value of the four values. Artif. Intell. 102, 97–141 (1998)
14. Arieli, O., Denecker, M.: Reducing preferential paraconsistent reasoning to classical entailment. J. Log. Comput. 13, 557–580 (2003)
15. Marquis, P., Porquet, N.: Computational aspects of quasi-classical entailment. Journal of Applied Non-Classical Logics 11, 295–312 (2001)
16. Schaerf, M., Cadoli, M.: Tractable reasoning via approximation. Artif. Intell. 74, 249–310 (1995)
17. Ma, Y., Qi, G., Hitzler, P., Lin, Z.: Measuring inconsistency for description logics based on paraconsistent semantics. In: ECSQARU. LNCS (LNAI), vol. 4724, pp. 30–41. Springer, Heidelberg (2007)