Towards a Systematic Account of Different Logic Programming Semantics

Pascal Hitzler

Department of Computer Science, Dresden University of Technology
www.wv.inf.tu-dresden.de/~pascal/
phitzler@inf.tu-dresden.de

Abstract. In [1, 2], a new methodology has been proposed which allows to derive uniform characterizations of different declarative semantics for logic programs with negation. One result from this work is that the well-founded semantics can formally be understood as a stratified version of the Fitting (or Kripke-Kleene) semantics. The constructions leading to this result, however, show a certain asymmetry which is not readily understood. We will study this situation here with the result that we will obtain a coherent picture of relations between different semantics.

1 Introduction

Within the past twenty years, many different declarative semantics for logic programs with negation have been developed. Different perspectives on the question what properties a semantics should foremost satisfy, have led to a variety of diverse proposals. From a knowledge representation and reasoning point of view it appears to be important that a semantics captures established non-monotonic reasoning frameworks, e.g. Reiters default logic [3], and that they allow intuitively appealing, i.e. "common sense", encodings of AI problems. The semantics which, due to common opinion by researchers in the field, satisfy these requirements best, are the least model semantics for definite programs [4], and for normal programs the stable [5] and the well-founded semantics [6]. Of lesser importance, albeit still acknowledged in particular for their relation to resolution-based logic programming, are the Fitting semantics [7] and approaches based on stratification [8, 9].

The semantics just mentioned are closely connnected by a number of well- (and some lesser-) known relationships, and many authors have contributed to this understanding. Fitting [10] provides a framework using Belnap's four-valued logic which encompasses supported, stable, Fitting, and well-founded semantics. His work was recently extended by Denecker, Marek, and Truszczynski [11]. Przymusinsky [12] gives a version in three-valued logic of the stable semantics, and shows that it coincides with the well-founded one. Van Gelder [13] constructs the well-founded semantics unsing the Gelfond-Lifschitz operator originally associated with the stable semantics. Dung and Kanchanasut [14] define the notion of fixpoint completion of a program which provides connections between the supported and the stable semantics, as well as between the Fitting and the well-founded semantics, studied by Fages [15] and Wendt [16]. Hitzler and Wendt [1, 2] have recently provided a unifying framework using level mappings, and results which amongst other things give further support to the point of view that the stable semantics is a formal and natural extension to normal programs of the least model semantics for definite programs. Furthermore, it was shown that the well-founded semantics can be understood, formally, as a stratified version of the Fitting semantics.

This latter result, however, exposes a certain asymmetry in the construction leading to it, and it is natural to ask the question as to what exactly is underlying it. This is what we will study in the sequel. In a nutshell, we will see that formally this asymmetry is due to the well-known preference of falsehood in logic programming semantics. More importantly, we will also see that a "dual" theory, obtained from prefering truth, can be stablished which is in perfect analogy to the close and well-known relationships between the different semantics mentioned above. We want to make it explicit from the start that we do not intend to provide new semantics for practical purposes[1]. We rather want to focus on the deepening of the theoretical insights into the relations between different semantics, by painting a coherent and complete picture of the dependencies and interconnections. We find the richness of the theory very appealing, and strongly supportive of the opinion that the major semantics studied in the field are founded on a sound theoretical base.

The plan of the paper is as follows. In Section 2 we will introduce notation and terminology needed for proving the results in the main body of the paper. We will also review in detail those results from [1, 2] which triggered and motivated our investigations. In Section 3 we will provide a variant of the stable semantics which prefers truth, and in Section 4 we will do likewise for the well-founded semantics. Throughout, our definitions will be accompanied by results which complete the picture of relationships between different semantics.

2 Preliminaries and Notation

A (*normal*) *logic program* is a finite set of (universally quantified) *clauses* of the form $\forall(A \leftarrow A_1 \wedge \cdots \wedge A_n \wedge \neg B_1 \wedge \cdots \wedge \neg B_m)$, commonly written as $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$, where A, A_i, and B_j, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, are atoms over some given first order language. A is called the *head* of the clause, while the remaining atoms make up the *body* of the clause, and depending on context, a body of a clause will be a set of literals (i.e. atoms or negated atoms) or the conjunction of these literals. Care will be taken that this identification does not cause confusion. We allow a body, i.e. a conjunction, to be empty, in which case it always evaluates to true. A clause with empty body is called a *unit clause* or a *fact*. A clause is called *definite*, if it contains no negation symbol. A program is called *definite* if it consists only of definite

[1] Although there may be some virtue to this perspective, see [17].

clauses. We will usually denote atoms with A or B, and literals, which may be atoms or negated atoms, by L or K.

Given a logic program P, we can extract from it the components of a first order language, and we always make the mild assumption that this language contains at least one constant symbol. The corresponding set of ground atoms, i.e. the *Herbrand base* of the program, will be denoted by B_P. For a subset $I \subseteq B_P$, we set $\neg I = \{\neg A \mid A \in B_P\}$. The set of all ground instances of P with respect to B_P will be denoted by $\mathsf{ground}(P)$. For $I \subseteq B_P \cup \neg B_P$, we say that A is *true with respect to* (or *in*) I if $A \in I$, we say that A is *false with respect to* (or *in*) I if $\neg A \in I$, and if neither is the case, we say that A is *undefined with respect to* (or *in*) I. A (*three-valued* or *partial*) *interpretation* I for P is a subset of $B_P \cup \neg B_P$ which is *consistent*, i.e. whenever $A \in I$ then $\neg A \notin I$. A body, i.e. a conjunction of literals, is true in an interpretation I if every literal in the body is true in I, it is false in I if one of its literals is false in I, and otherwise it is undefined in I. For a negated literal $L = \neg A$ we will find it convenient to write $\neg L \in I$ if $A \in I$. By I_P we denote the set of all (three-valued) interpretations of P. Both I_P and $B_P \cup \neg B_P$ are complete partial orders (cpos) via set-inclusion, i.e. they contain the empty set as least element, and every ascending chain has a supremum, namely its union. A *model* of P is an interpretation $I \in I_P$ such that for each clause $A \leftarrow \mathsf{body}$ we have that $\mathsf{body} \subseteq I$ implies $A \in I$. A *total* interpretation is an interpretation I such that no $A \in B_P$ is undefined in I.

For an interpretation I and a program P, an *I-partial level mapping* for P is a partial mapping $l : B_P \to \alpha$ with domain $\mathsf{dom}(l) = \{A \mid A \in I \text{ or } \neg A \in I\}$, where α is some (countable) ordinal. We extend every level mapping to literals by setting $l(\neg A) = l(A)$ for all $A \in \mathsf{dom}(l)$. A (*total*) *level mapping* is a total mapping $l : B_P \to \alpha$ for some (countable) ordinal α.

Given a normal logic program P and some $I \subseteq B_P \cup \neg B_P$, we say that $U \subseteq B_P$ is an *unfounded set* (*of* P) *with respect to* I if each atom $A \in U$ satisfies the following condition: For each clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ (at least) one of the following holds.

(Ui) Some (positive or negative) literal in body is false in I.
(Uii) Some (non-negated) atom in body occurs in U.

Given a normal logic program P, we define the following operators on $B_P \cup \neg B_P$. $T_P(I)$ is the set of all $A \in B_P$ such that there exists a clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ such that body is true in I. $F_P(I)$ is the set of all $A \in B_P$ such that for all clauses $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ we have that body is false in I. Both T_P and F_P map elements of I_P to elements of I_P. Now define the operator $\Phi_P : I_P \to I_P$ by

$$\Phi_P(I) = T_P(I) \cup \neg F_P(I).$$

This operator is due to [7] and is well-defined and monotonic on the cpo I_P, hence has a least fixed point by the Knaster-Tarski[2] fixed-point theorem, and we can obtain this fixed point by defining, for each monotonic operator F, that

[2] We follow the terminology from [18]. The Knaster-Tarski theorem is sometimes called Tarski theorem and states that every monotonic function on a cpo has a least fixed

$F\uparrow 0 = \emptyset$, $F\uparrow(\alpha+1) = F(F\uparrow\alpha)$ for any ordinal α, and $F\uparrow\beta = \bigcup_{\gamma<\beta} F\uparrow\gamma$ for any limit ordinal β, and the least fixed point of F is obtained as $F\uparrow\alpha$ for some ordinal α. The least fixed point of Φ_P is called the *Kripke-Kleene model* or *Fitting model* of P, determining the *Fitting semantics* of P.

Now, for $I \subseteq B_P \cup \neg B_P$, let $U_P(I)$ be the greatest unfounded set (of P) with respect to I, which always exists due to [6]. Finally, define

$$W_P(I) = T_P(I) \cup \neg U_P(I)$$

for all $I \subseteq B_P \cup \neg B_P$. The operator W_P, which operates on the cpo $B_P \cup \neg B_P$, is due to [6] and is monotonic, hence has a least fixed point by the Knaster-Tarski[2] fixed-point theorem, as above for Φ_P. It turns out that $W_P\uparrow\alpha$ is in I_P for each ordinal α, and so the least fixed point of W_P is also in I_P and is called the *well-founded model* of P, giving the *well-founded semantics* of P.

In order to avoid confusion, we will use the following terminology: the notion of *interpretation*, and I_P will be the set of all those, will by default denote consistent subsets of $B_P \cup \neg B_P$, i.e. interpretations in three-valued logic. We will sometimes emphasize this point by using the notion *partial interpretation*. By *two-valued interpretations* we mean subsets of B_P. Both interpretations and two-valued interpretations are ordered by subset inclusion. Each two-valued interpretation I can be identified with the partial interpretation $I' = I \cup \neg(B_P \setminus I)$. Note however, that in this case I' is always a maximal element in the ordering for partial interpretations, while I is in general not maximal as a two-valued interpretation[3]. Given a partial interpretation I, we set $I^+ = I \cap B_P$ and $I^- = \{A \in B_P \mid \neg A \in I\}$.

Given a program P, we define the operator T_P^+ on subsets of B_P by $T_P^+(I) = T_P(I \cup \neg(B_P \setminus I))$. The pre-fixed points of T_P^+, i.e. the two-valued interpretations $I \subseteq B_P$ with $T_P^+(I) \subseteq I$, are exactly the models, in the sense of classical logic, of P. Post-fixed points of T_P^+, i.e. $I \subseteq B_P$ with $I \subseteq T_P^+(I)$ are called *supported interpretations* of P, and a supported model of P is a model P which is a supported interpretation. The supported models of P thus coincide with the fixed points of T_P^+. It is well-known that for definite programs P the operator T_P^+ is monotonic on the set of all subsets of B_P, with respect to subset inclusion. Indeed it is Scott-continuous [4, 20] and, via the Tarski-Kantorovich[2] fixed-point theorem, achieves its least pre-fixed point M, which is also a fixed point, as the supremum of the iterates $T_P^+\uparrow n$ for $n \in \mathbb{N}$. So $M = \mathsf{lfp}\left(T_P^+\right) = T_P^+\uparrow\omega$ is *the least two-valued model* of P. Likewise, since the set of all subsets of B_P is

point, which can be obtained by transfinitely iterating the bottom element of the cpo. The Tarski-Kantorovitch theorem is sometimes refered to as the Kleene theorem or the Scott theorem (or even as "the" fixed-point theorem) and states that if the function is additionally Scott (or order-) continuous, then the least fixed point can be obtained by an iteration which is not transfinite, i.e. closes off at ω, the least infinite ordinal. In both cases, the least fixed point is also the least pre-fixed point of the function.

[3] These two orderings in fact correspond to the knowledge and truth orderings as discussed in [19].

a complete lattice, and therefore has greatest element B_P, we can also define $T_P^+ \downarrow 0 = B_P$ and inductively $T_P^+ \downarrow (\alpha + 1) = T_P^+(T_P^+ \downarrow \alpha)$ for each ordinal α and $T_P^+ \downarrow \beta = \bigcap_{\gamma<\beta} T_P^+ \downarrow \gamma$ for each limit ordinal β. Again by the Knaster-Tarski fixed-point theorem, applied to the superset inclusion ordering (i.e. reverse subset inclusion) on subsets of B_P, it turns out that T_P^+ has a greatest fixed point, $\mathsf{gfp}\left(T_P^+\right)$.

The stable model semantics due to [5] is intimately related to the well-founded semantics. Let P be a normal program, and let $M \subseteq B_P$ be a set of atoms. Then we define P/M to be the (ground) program consisting of all clauses $A \leftarrow A_1, \ldots, A_n$ for which there is a clause $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(P)$ with $B_1, \ldots, B_m \notin M$. Since P/M does no longer contain negation, it has a least two-valued model $T_{P/M}^+ \uparrow \omega$. For any two-valued interpretation I we can therefore define the operator $\mathrm{GL}_P(I) = T_{P/I}^+ \uparrow \omega$, and call M a *stable model* of the normal program P if it is a fixed point of the operator GL_P, i.e. if $M = \mathrm{GL}_P(M) = T_{P/M}^+ \uparrow \omega$. As it turns out, the operator GL_P is in general not monotonic for normal programs P. However it is *antitonic*, i.e. whenever $I \subseteq J \subseteq B_P$ then $\mathrm{GL}_P(J) \subseteq \mathrm{GL}_P(I)$. As a consequence, the operator GL_P^2, obtained by applying GL_P twice, is monotonic, and hence has a least fixed point L_P and a greatest fixed point G_P. In [13] it was shown that $\mathrm{GL}_P(L_P) = G_P$, $L_P = \mathrm{GL}_P(G_P)$, and that $L_P \cup \neg(B_P \setminus G_P)$ coincides with the well-founded model of P. This is called the *alternating fixed point characterization* of the well-founded semantics.

Some Results

The following is a straightforward result which has, to the best of our knowledge, not been noted before. It follows the general approach put forward in [1, 2].

Theorem 1. *Let P be a definite program. Then there is a unique two-valued model M of P for which there exists a (total) level mapping $l : B_P \to \alpha$ such that for each atom $A \in M$ there exists a clause $A \leftarrow A_1, \ldots, A_n$ in* $\mathsf{ground}(P)$ *with $A_i \in M$ and $l(A) > l(A_i)$ for all $i = 1, \ldots, n$. Furthermore, M is the least two-valued model of P.*

Proof. Let M be the least two-valued model $T_P^+ \uparrow \omega$, choose $\alpha = \omega$, and define $l : B_P \to \alpha$ by setting $l(A) = \min\{n \mid A \in T_P^+ \uparrow (n+1)\}$, if $A \in M$, and by setting $l(A) = 0$, if $A \notin M$. From the fact that $\emptyset \subseteq T_P^+ \uparrow 1 \subseteq \ldots \subseteq T_P^+ \uparrow n \subseteq \ldots \subseteq T_P^+ \uparrow \omega = \bigcup_m T_P^+ \uparrow m$, for each n, we see that l is well-defined and that the least model $T_P^+ \uparrow \omega$ for P has the desired properties.

Conversely, if M is a two-valued model for P which satisfies the given condition for some mapping $l : B_P \to \alpha$, then it is easy to show, by induction on $l(A)$, that $A \in M$ implies $A \in T_P^+ \uparrow (l(A) + 1)$. This yields that $M \subseteq T_P^+ \uparrow \omega$, and hence that $M = T_P^+ \uparrow \omega$ by minimality of the model $T_P^+ \uparrow \omega$.

The following result is due to [15], and is striking in its similarity to Theorem 1.

Theorem 2. *Let P be normal. Then a two-valued model $M \subseteq B_P$ of P is a stable model of P if and only if there exists a (total) level mapping $l : B_P \to \alpha$ such that for each $A \in M$ there exists $A \leftarrow A_1, \ldots, A_n \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(P)$ with $A_i \in M$, $B_j \notin M$, and $l(A) > l(A_i)$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.*

We next recall the following alternative characterization of the Fitting model, due to [1, 2].

Definition 1. *Let P be a normal logic program, I be a model of P, and l be an I-partial level mapping for P. We say that P* satisfies *(F)* with respect to *I* and *l, if each $A \in \mathsf{dom}(l)$ satisfies one of the following conditions.*

(Fi) $A \in I$ *and there exists a clause* $A \leftarrow L_1, \ldots, L_n$ *in* $\mathsf{ground}(P)$ *such that* $L_i \in I$ *and* $l(A) > l(L_i)$ *for all* i.

(Fii) $\neg A \in I$ *and for each clause* $A \leftarrow L_1, \ldots, L_n$ *in* $\mathsf{ground}(P)$ *there exists* i *with* $\neg L_i \in I$ *and* $l(A) > l(L_i)$.

Theorem 3. *Let P be a normal logic program with Fitting model M. Then M is the greatest model among all models I, for which there exists an I-partial level mapping l for P such that P satisfies (F) with respect to I and l.*

Let us recall next the definition of a (locally) stratified program, due to [8, 9]: A normal logic program is called *locally stratified* if there exists a (total) level mapping $l : B_P \to \alpha$, for some ordinal α, such that for each clause $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(P)$ we have that $l(A) \geq l(A_i)$ and $l(A) > l(B_j)$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The notion of (locally) stratifed program was developed with the idea of preventing *recursion through negation*, while allowing recursion through positive dependencies. (Locally) stratified programs have total well-founded models.

There exist locally stratified programs which do not have a total Fitting semantics and vice versa — just consider the programs consisting of the single clauses $p \leftarrow p$, respectively, $p \leftarrow \neg p, q$. In fact, condition (Fii) requires a strict decrease of level between the head and a literal in the rule, independent of this literal being positive or negative. But, on the other hand, condition (Fii) imposes no further restrictions on the remaining body literals, while the notion of local stratification does. These considerations motivate the substitution of condition (Fii) by the condition (Cii), as done for the following definition.

Definition 2. *Let P be a normal logic program, I be a model of P, and l be an I-partial level mapping for P. We say that P* satisfies *(WF)* with respect to *I* and *l, if each $A \in \mathsf{dom}(l)$ satisfies (Fi) or the following condition.*

(Cii) $\neg A \in I$ *and for each clause* $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ *contained in* $\mathsf{ground}(P)$ *(at least) one of the following conditions holds:*
(Ciia) There exists $i \in \{1, \ldots, n\}$ *with* $\neg A_i \in I$ *and* $l(A) \geq l(A_i)$.
(Ciib) There exists $j \in \{1, \ldots, m\}$ *with* $B_j \in I$ *and* $l(A) > l(B_j)$.

So, in the light of Theorem 3, Definition 2 should provide a natural "stratified version" of the Fitting semantics. And indeed it does, and furthermore, the resulting semantics coincides with the well-founded semantics, which is a very satisfactory result from [1,2].

Theorem 4. *Let P be a normal logic program with well-founded model M. Then M is the greatest model among all models I, for which there exists an I-partial level mapping l for P such that P satisfies (WF) with respect to I and l.*

For completeness, we remark that an alternative characterization of the weakly perfect model semantics [21] can also be found in [1,2].

The approach which led to the results just mentioned, originally put forward in [1,2], provides a methodology for obtaining uniform characterizations of different semantics for logic programs.

3 Maximally Circular Stable Semantics

We note that condition (Fi) has been reused in Definition 2. Thus, Definition 1 has been "stratified" only with respect to condition (Fii), yielding (Cii), but not with respect to (Fi). Indeed, also replacing (Fi) by a stratified version such as the following seems not satisfactory at first sight.

(Ci) $A \in I$ and there exists a clause $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ in $\mathsf{ground}(P)$ such that $A_i, \neg B_j \in I$, $l(A) \geq l(A_i)$, and $l(A) > l(B_j)$ for all i and j.

If we replace condition (Fi) by condition (Ci) in Definition 2, then it is not guaranteed that for any given program there is a greatest model satisfying the desired properties, as the following example from [1,2] shows.

Example 1. Consider the program consisting of the two clauses $p \leftarrow p$ and $q \leftarrow \neg p$, and the two (total) models $M_1 = \{p, \neg q\}$ and $M_2 = \{\neg p, q\}$, which are incomparable, and the level mapping l with $l(p) = 0$ and $l(q) = 1$.

In order to arrive at an understanding of this asymmetry, we consider the setting with conditions (Ci) and (Fii), which is somehow "dual" to the well-founded semantics which is characterized by (Fi) and (Cii).

Definition 3. *Let P be a normal logic program, I be a model of P, and l be an I-partial level mapping for P. We say that P* satisfies *(CW)* with respect to *I* and *l, if each $A \in \mathsf{dom}(l)$ satisfies (Ci) or (Fii).*

By virtue of Definition 3 we will be able to develop a theory which complements the restults from Section 2. We will first characterize the greatest model of a definite program analogously to Theorem 1.

Theorem 5. *Let P be a definite program. Then there is a unique two-valued supported interpretation M of P for which there exists a (total) level mapping $l : B_P \to \alpha$ such that for each atom $A \notin M$ and for all clauses $A \leftarrow A_1, \ldots, A_n$ in $\mathsf{ground}(P)$ there is some $A_i \notin M$ with $l(A) > l(A_i)$. Furthermore, M is the greatest two-valued model of P.*

Proof. Let M be the greatest two-valued model of P, and let α be the least ordinal such that $M = T_P^+ \downarrow \alpha$. Define $l : B_P \to \alpha$ by setting $l(A) = \min\{\gamma \mid A \notin T_P^+ \downarrow (\gamma + 1)\}$ for $A \notin M$, and by setting $l(A) = 0$ if $A \in M$. The mapping l is well-defined because $A \notin M$ with $A \notin T_P^+ \downarrow \gamma = \bigcap_{\beta<\gamma} T_P^+ \downarrow \beta$ for some limit ordinal γ implies $A \notin T_P^+ \downarrow \beta$ for some $\beta < \gamma$. So the least ordinal β with $A \notin T_P^+ \downarrow \beta$ is always a successor ordinal. Now assume that there is $A \notin M$ which does not satisfy the stated condition. We can furthermore assume without loss of generality that A is chosen with this property such that $l(A)$ is minimal. Let $A \leftarrow A_1, \ldots, A_n$ be a clause in $\mathsf{ground}(P)$. Since $A \notin T_P^+ \left(T_P^+ \downarrow l(A)\right)$ we obtain $A_i \notin T_P^+ \downarrow l(A) \supseteq M$ for some i. But then $l(A_i) < l(A)$ which contradicts minimality of $l(A)$.

Conversely, let M be a two-valued model for P which satisfies the given condition for some mapping $l : B_P \to \alpha$. We show by transfinite induction on $l(A)$ that $A \notin M$ implies $A \notin T_P^+ \downarrow (l(A) + 1)$, which suffices because it implies that for the greatest two-valued model $T_P^+ \downarrow \beta$ of P we have that $T_P^+ \downarrow \beta \subseteq M$, and therefore $T_P^+ \downarrow \beta = M$. For the inductive proof consider first the case where $l(A) = 0$. Then there is no clause in $\mathsf{ground}(P)$ with head A and consequently $A \notin T_P^+ \downarrow 1 = T_P^+(B_P)$. Now assume that the statement to be proven holds for all $B \notin M$ with $l(B) < \alpha$, where α is some ordinal, and let $A \notin M$ with $l(A) = \alpha$. Then each clause in $\mathsf{ground}(P)$ with head A contains an atom B with $l(B) = \beta < \alpha$ and $B \notin M$. Hence $B \notin T_P^+ \downarrow (\beta + 1)$ and consequently $A \notin T_P^+ \downarrow (\alpha + 1)$.

The following definition and theorem are analogous to Theorem 2.

Definition 4. *Let P be normal. Then $M \subseteq B_P$ is called a* maximally circular stable model *(*maxstable model*) of P if it is a two-valued supported interpretation of P and there exists a (total) level mapping $l : B_P \to \alpha$ such that for each atom $A \notin M$ and for all clauses $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(P)$ with $B_1, \ldots, B_m \notin M$ there is some $A_i \notin M$ with $l(A) > l(A_i)$.*

Theorem 6. *$M \subseteq B_P$ is a maxstable model of P if and only if $M = \mathsf{gfp}\left(T_{P/M}^+\right)$.*

Proof. First note that every maxstable model is a a supported model. Indeed supportedness follows immediately from the definition. Now assume that M is maxstable but is not a model, i.e. there is $A \notin M$ but there is a clause $A \leftarrow A_1, \ldots, A_n$ in $\mathsf{ground}(P)$ with $A_i \in M$ for all i. But by the definition of maxstable model we must have that there is $A_i \notin M$, which contradicts $A_i \in M$.

Now let M be a maxstable model of P. Let $A \notin M$ and let $T_{P/M}^+ \downarrow \alpha = \mathsf{gfp}\left(T_{P/M}^+\right)$. We show by transfinite induction on $l(A)$ that $A \notin T_{P/M}^+ \downarrow (l(A)+1)$

and hence $A \notin T^+_{P/M}\!\downarrow\!\alpha$. For $l(A) = 0$ there is no clause with head A in P/M, so $A \notin T^+_{P/M}\!\downarrow\!1$. Now let $l(A) = \beta$ for some ordinal β. By assumption we have that for all clauses $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ with $B_1, \dots, B_m \notin M$ there exists $A_i \notin M$ with $l(A) > l(A_i)$, say $l(A_i) = \gamma < \beta$. Hence $A_i \notin T^+_{P/M}\!\downarrow\!(\gamma + 1)$, and consequently $A \notin T^+_{P/M}\!\downarrow\!(\beta + 1)$, which shows that $\mathsf{gfp}\left(T^+_{P/M}\right) \subseteq M$.

So let again M be a maxstable model of P and let $A \notin \mathsf{gfp}\left(T^+_{P/M}\right) = T^+_{P/M}\!\downarrow\!\alpha$ and $l(A) = \beta$. Then for each clause $A \leftarrow A_1, \dots, A_n$ in P/M there is A_i with $A_i \notin T^+_{P/M}\!\downarrow\!\alpha$ and $l(A) > l(A_i)$. Now assume $A \in M$. Without loss of generality we can furthermore assume that A is chosen such that $l(A) = \beta$ is minimal. Hence $A_i \notin M$, and we obtain that for each clause in P/M with head A one of the corresponding body atoms is false in M. By supportedness of M this yields $A \notin M$, which contradicts our assumption. Hence $A \notin M$ as desired.

Conversely, let $M = \mathsf{gfp}\left(T^+_{P/M}\right)$. Then as an immediate consequence of Theorem 5 we obtain that M is maxstable.

4 Maximally Circular Well-Founded Semantics

Maxstable models are formally analogous[4] to stable models in that the former are fixed points of the operator $I \mapsto \mathsf{gfp}\left(T^+_{P/I}\right)$, while the latter are fixed points of the operator $I \mapsto \mathsf{lfp}\left(T^+_{P/I}\right)$. Further, in analogy to the alternating fixed point characterization of the well-founded model, we can obtain a corresponding variant of the well-founded semantics, which we will do next. Theorem 6 suggests the defininition of the following operator.

Definition 5. *Let P be a normal program and I be a two-valued interpretation. Then define* $\mathrm{CGL}_P(I) = \mathsf{gfp}\left(T^+_{P/I}\right)$.

Using the operator CGL_P, we can define a "maximally circular" version of the alternating fixed-point semantics.

Proposition 1. *Let P be a normal program. Then the following hold.*

(i) CGL_P *is antitonic and* CGL^2_P *is monotonic.*
(ii) $\mathrm{CGL}_P\left(\mathsf{lfp}\left(\mathrm{CGL}^2_P\right)\right) = \mathsf{gfp}\left(\mathrm{CGL}^2_P\right)$ *and* $\mathrm{CGL}_P\left(\mathsf{gfp}\left(\mathrm{CGL}^2_P\right)\right) = \mathsf{lfp}\left(\mathrm{CGL}^2_P\right)$.

Proof. (i) If $I \subseteq J \in B_P$, then $P/J \subseteq P/I$ and consequently $\mathrm{CGL}_P(J) = \mathsf{gfp}\left(T^+_{P/J}\right) \subseteq \mathsf{gfp}\left(T^+_{P/I}\right) = \mathrm{CGL}_P(I)$. Monotonicity of CGL^2_P then follows trivially.

(ii) Let $L_P = \mathsf{lfp}\left(\mathrm{CGL}^2_P\right)$ and $G_P = \mathsf{gfp}\left(\mathrm{CGL}^2_P\right)$. Then we can calculate $\mathrm{CGL}^2_P(\mathrm{CGL}_P(L_P)) = \mathrm{CGL}_P\left(\mathrm{CGL}^2_P(L_P)\right) = \mathrm{CGL}_P(L_P)$, so $\mathrm{CGL}_P(L_P)$ is a

[4] The term *dual* seems not to be entirely adequate in this situation, although it is intuitionally appealing.

fixed point of CGL_P^2, and hence $L_P \subseteq \mathrm{CGL}_P(L_P) \subseteq G_P$. Similarly, $L_P \subseteq \mathrm{CGL}_P(G_P) \subseteq G_P$. Since $L_P \subseteq G_P$ we get from the antitonicity of CGL_P that $L_P \subseteq \mathrm{CGL}_P(G_P) \subseteq \mathrm{CGL}_P(L_P) \subseteq G_P$. Similarly, since $\mathrm{CGL}_P(L_P) \subseteq G_P$, we obtain $\mathrm{CGL}_P(G_P) \subseteq \mathrm{CGL}_P^2(L_P) = L_P \subseteq \mathrm{CGL}_P(G_P)$, so $\mathrm{CGL}_P(G_P) = L_P$, and also $G_P = \mathrm{CGL}_P^2(G_P) = \mathrm{CGL}_P(L_P)$.

We will now define an operator for the maximally circular well-founded semantics. Given a normal logic program P and some $I \in I_P$, we say that $S \subseteq B_P$ is a *self-founded set (of P) with respect to I* if $S \cup I \in I_P$ and each atom $A \in S$ satisfies the following condition: There exists a clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ such that one of the following holds.

(Si) body is true in I.
(Sii) Some (non-negated) atoms in body occur in S and all other literals in body are true in I.

Self-founded sets are analogous[5] to unfounded sets, and the following proposition holds.

Proposition 2. *Let P be a normal program and let $I \in I_P$. Then there exists a greatest self-founded set of P with respect to I.*

Proof. If $(S_i)_{i\in\mathcal{I}}$ is a family of sets each of which is a self-founded set of P with respect to I, then it is easy to see that $\bigcup_{i\in\mathcal{I}} S_i$ is also a self-founded set of P with respect to I.

Given a normal program P and $I \in I_P$, let $S_P(I)$ be the greatest self-founded set of P with respect to I, and define the operator CW_P on I_P by

$$\mathrm{CW}_P(I) = S_P(I) \cup \neg F_P(I).$$

Proposition 3. *The operator CW_P is well-defined and monotonic.*

Proof. For well-definedness, we have to show that $S_P(I) \cap F_P(I) = \emptyset$ for all $I \in I_P$. So assume there is $A \in S_P(I) \cap F_P(I)$. From $A \in F_P(I)$ we obtain that for each clause with head A there is a corresponding body literal L which is false in I. From $A \in S_P(I)$, more precisely from (Sii), we can furthermore conclude that L is an atom and $L \in S_P(I)$. But then $\neg L \in I$ and $L \in S_P(I)$ which is impossible by definition of self-founded set which requires that $S_P(I) \cup I \in I_P$. So $S_P(I) \cap F_P(I) = \emptyset$ and CW_P is well-defined.

For monotonicity, let $I \subseteq J \in I_P$ and let $L \in \mathrm{CW}_P(I)$. If $L = \neg A$ is a negated atom, then $A \in F_P(I)$ and all clauses with head A contain a body literal which is false in I, hence in J, and we obtain $A \in F_P(J)$. If $L = A$ is an atom, then $A \in S_P(I)$ and there exists a clause $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ such that (at least) one of (Si) or (Sii) holds. If (Si) holds, then body is true in I, hence in J, and $A \in S_P(J)$. If (Sii) holds, then some non-negated atoms in body occur in S and all other literals in body are true in I, hence in J, and we obtain $A \in S_P(J)$.

[5] Again, it is not really a duality.

The following theorem relates our previous observations to Definition 3, in perfect analogy to the correspondence between the stable model semantics, Theorem 1, Fages's characterization from Theorem 2, the well-founded semantics, and the alternating fixed point characterization.

Theorem 7. *Let P be a normal program and $M_P = \mathsf{lfp}(\mathrm{CW}_P)$. Then the following hold.*

(i) M_P is the greatest model among all models I of P such that there is an I-partial level mapping l for P such that P satisfies (CW) with respect to I and l.
(ii) $M_P = \mathsf{lfp}\left(\mathrm{CGL}_P^2\right) \cup \neg\left(B_P \setminus \mathsf{gfp}\left(\mathrm{CGL}_P^2\right)\right)$.

Proof. (i) Let $M_P = \mathsf{lfp}(\mathrm{CW}_P)$ and define the M_P-partial level mapping l_P as follows: $l_P(A) = \alpha$, where α is the least ordinal such that A is not undefined in $\mathrm{CW}_P\!\uparrow\!(\alpha+1)$. The proof will be established by showing the following facts: (1) P satisfies (CW) with respect to M_P and l_P. (2) If I is a model of P and l is an I-partial level mapping such that P satisfies (CW) with respect to I and l, then $I \subseteq M_P$.

(1) Let $A \in \mathsf{dom}(l_P)$ and $l_P(A) = \alpha$. We consider two cases.

(Case i) If $A \in M_P$, then $A \in S_P(\mathrm{CW}_P\uparrow\alpha)$, hence there exists a clause $A \leftarrow \mathtt{body}$ in $\mathsf{ground}(P)$ such that (Si) or (Sii) holds with respect to $\mathrm{CW}_P\!\uparrow\!\alpha$. If (Si) holds, then all literals in $\mathtt{body}$ are true in $\mathrm{CW}_P\!\uparrow\!\alpha$, hence have level less than $l_P(A)$ and (Ci) is satisfied. If (Sii) holds, then some non-negated atoms from $\mathtt{body}$ occur in $S_P(\mathrm{CW}_P\!\uparrow\!\alpha)$, hence have level less than or equal to $l_P(A)$, and all remaining literals in $\mathtt{body}$ are true in $\mathrm{CW}_P\!\uparrow\!\alpha$, hence have level less than $l_P(A)$. Consequently, A satisfies (Ci) with respect to M_P and l_P.

(Case ii) If $\neg A \in M_P$, then $A \in F_P(\mathrm{CW}_P\!\uparrow\!\alpha)$, hence for all clauses $A \leftarrow \mathtt{body}$ in $\mathsf{ground}(P)$ there exists $L \in \mathtt{body}$ with $\neg L \in \mathrm{CW}_P\!\uparrow\!\alpha$ and $l_P(L) < \alpha$, hence $\neg L \in M_P$. Consequently, A satisfies (Fii) with respect to M_P and l_P, and we have established that fact (1) holds.

(2) We show via transfinite induction on $\alpha = l(A)$, that whenever $A \in I$ (respectively, $\neg A \in I$), then $A \in \mathrm{CW}_P\uparrow(\alpha+1)$ (respectively, $\neg A \in \mathrm{CW}_P\uparrow(\alpha+1)$). For the base case, note that if $l(A) = 0$, then $\neg A \in I$ implies that there is no clause with head A in $\mathsf{ground}(P)$, hence $\neg A \in \mathrm{CW}_P\uparrow 1$. If $A \in I$ then consider the set S of all atoms B with $l(B) = 0$ and $B \in I$. We show that S is a self-founded set of P with respect to $\mathrm{CW}_P\!\uparrow\!0 = \emptyset$, and this suffices since it implies $A \in \mathrm{CW}_P\!\uparrow\!1$ by the fact that $A \in S$. So let $C \in S$. Then $C \in I$ and C satisfies condition (Ci) with respect to I and l, and since $l(C) = 0$, we have that there is a definite clause with head C whose body atoms (if it has any) are all of level 0 and contained in I. Hence condition (Sii) (or (Si)) is satisfied for this clause and S is a self-founded set of P with respect to I. So assume now that the induction hypothesis holds for all $B \in B_P$ with $l(B) < \alpha$, and let A be such that $l(A) = \alpha$. We consider two cases.

(Case i) If $A \in I$, consider the set S of all atoms B with $l(B) = \alpha$ and $B \in I$. We show that S is a self-founded set of P with respect to $\mathrm{CW}_P\uparrow\alpha$, and this suffices since it implies $A \in \mathrm{CW}_P\uparrow(\alpha+1)$ by the fact that $A \in S$. First note that $S \subseteq I$, so $S \cup I \in I_P$. Now let $C \in S$. Then $C \in I$ and C satisfies condition (Ci) with respect to I and l, so there is a clause $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ in $\mathsf{ground}(P)$ such that $A_i, \neg B_j \in I$, $l(A) \geq l(A_i)$, and $l(A) > l(B_j)$ for all i and j. By induction hypothesis we obtain $\neg B_j \in \mathrm{CW}_P\!\uparrow\!\alpha$. If $l(A_i) < l(A)$ for some A_i then we have $A_i \in \mathrm{CW}_P\!\uparrow\!\alpha$, also by induction hypothesis. If there is no A_i with $l(A_i) = l(A)$, then (Si) holds, while $l(A_i) = l(A)$ implies $A_i \in S$, so (Sii) holds.

(Case ii) If $\neg A \in I$, then A satisfies (Fii) with respect to I and l. Hence for all clauses $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ we have that there is $L \in \mathsf{body}$ with $\neg L \in I$ and $l(L) < \alpha$. Hence for all these L we have $\neg L \in \mathrm{CW}_P\!\uparrow\!\alpha$ by induction hypothesis, and consequently for all clauses $A \leftarrow \mathsf{body}$ in $\mathsf{ground}(P)$ we obtain that body is false in $\mathrm{CW}_P\!\uparrow\!\alpha$ which yields $\neg A \in \mathrm{CW}_P\!\uparrow\!(\alpha+1)$. This establishes fact (2) and concludes the proof of (i).

(ii) We first introduce some notation. Let

$$\begin{aligned} L_0 &= \emptyset, & G_0 &= B_P, \\ L_{\alpha+1} &= \mathrm{CGL}_P(G_\alpha), & G_{\alpha+1} &= \mathrm{CGL}_P(L_\alpha) \qquad \text{for any ordinal } \alpha, \\ L_\alpha &= \bigcup_{\beta<\alpha} L_\beta, & G_\alpha &= \bigcap_{\beta<\alpha} G_\beta \qquad \text{for limit ordinal } \alpha, \\ L_P &= \mathsf{lfp}(\mathrm{CGL}_P^2), & G_P &= \mathsf{gfp}(\mathrm{CGL}_P^2). \end{aligned}$$

By transfinite induction, it is easily checked that $L_\alpha \subseteq L_\beta \subseteq G_\beta \subseteq G_\alpha$ whenever $\alpha \leq \beta$. So $L_P = \bigcup L_\alpha$ and $G_P = \bigcap G_\alpha$.

Let $M = L_P \cup \neg(B_P \setminus G_P)$. We intend to apply (i) and first define an M-partial level mapping l. We will take as image set of l, pairs (α, γ) of ordinals, with the lexicographic ordering. This can be done without loss of generality since any set of such pairs, under the lexicographic ordering, is well-ordered, and therefore order-isomorphic to an ordinal. For $A \in L_P$, let $l(A)$ be the pair $(\alpha, 0)$, where α is the least ordinal such that $A \in L_{\alpha+1}$. For $B \notin G_P$, let $l(B)$ be the pair (β, γ), where β is the least ordinal such that $B \notin G_{\beta+1}$, and γ is least such that $B \notin T_{P/L_\beta}\!\downarrow\!\gamma$. It is easily shown that l is well-defined, and we show next by transfinite induction that P satisfies (CW) with respect to M and l.

Let $A \in L_1 = \mathsf{gfp}\left(T^+_{P/B_P}\right)$. Since P/B_P contains exactly all clauses from $\mathsf{ground}(P)$ which contain no negation, we have that A is contained in the greatest two-valued model of a definite subprogram of P, namely P/B_P. So there must be a definite clause in $\mathsf{ground}(P)$ with head A whose corresponding body atoms are also true in L_1, which, by definition of l, must have the same level as A, hence (Ci) is satisfied. Now let $\neg B \in \neg(B_P \setminus G_P)$ such that $B \in (B_P \setminus G_1) = B_P \setminus \mathsf{gfp}\left(T^+_{P/\emptyset}\right)$. Since $P/\emptyset$ contains all clauses from $\mathsf{ground}(P)$ with all negative literals removed, we obtain that B is not contained in the greatest two-valued model of the definite program $P/\emptyset$, and (Fii) is satisfied by Theorem 5 using a simple induction argument.

Assume now that, for some ordinal α, we have shown that A satisfies (CW) with respect to M and l for all $A \in B_P$ with $l(A) < (\alpha, 0)$.

Let $A \in L_{\alpha+1} \setminus L_\alpha = \mathsf{gfp}\left(T^+_{P/G_\alpha}\right) \setminus L_\alpha$. Then $A \in \left(T^+_{P/G_\alpha}\!\downarrow\!\gamma\right) \setminus L_\alpha$ for some γ; note that all (negative) literals which were removed by the Gelfond-Lifschitz transformation from clauses with head A have level less than $(\alpha, 0)$. Then A satisfies (Ci) with respect to M and l by definition of l.

Let $A \in (B_P \setminus G_{\alpha+1}) \cap G_\alpha$. Then $A \notin \mathsf{gfp}\left(T^+_{P/L_\alpha}\right)$ and we conclude again from Theorem 5, using a simple induction argument, that A satisfies (CW) with respect to M and l.

This finishes the proof that P satisfies (CW) with respect to M and l. It remains to show that M is greatest with this property.

So assume that $M_1 \supset M$ is the greatest model such that P satisfies (CW) with respect to M_1 and some M_1-partial level mapping l_1. Assume $L \in M_1 \setminus M$ and, without loss of generality, let the literal L be chosen such that $l_1(L)$ is minimal. We consider two cases.

(Case i) If $L = \neg A \in M_1 \setminus M$ is a negated atom, then by (Fii) for each clause $A \leftarrow L_1, \ldots, L_n$ in $\mathsf{ground}(P)$ there exists i with $\neg L_i \in M_1$ and $l_1(A) > l_1(L_i)$. Hence, $\neg L_i \in M$ and consequently for each clause $A \leftarrow \mathtt{body}$ in P/L_P we have that some atom in $\mathtt{body}$ is false in $M = L_P \cup \neg(B_P \setminus G_P)$. But then $A \notin \mathrm{CGL}_P(L_P) = G_P$, hence $\neg A \in M$, contradicting $\neg A \in M_1 \setminus M$.

(Case ii) If $L = A \in M_1 \setminus M$ is an atom, then $A \notin M = L_P \cup \neg(B_P \setminus G_P)$ and in particular $A \notin L_P = \mathsf{gfp}\left(T^+_{P/G_P}\right)$. Hence $A \notin T^+_{P/G_P}\!\downarrow\!\gamma$ for some γ, which can be chosen to be least with this property. We show by induction on γ that this leads to a contradiction, to finish the proof.

If $\gamma = 1$, then there is no clause with head A in P/G_P, i.e. for all clauses $A \leftarrow \mathtt{body}$ in $\mathsf{ground}(P)$ we have that $\mathtt{body}$ is false in M, hence in M_1, which contradicts $A \in M_1$.

Now assume that there is no $B \in M_1 \setminus M$ with $B \notin T^+_{P/G_P}\!\downarrow\!\delta$ for any $\delta < \gamma$, and let $A \in M_1 \setminus M$ with $A \notin T^+_{P/G_P}\!\downarrow\!\gamma$, which implies that γ is a successor ordinal. By $A \in M_1$ and (Ci) there must be a clause $A \leftarrow A_1, \ldots, A_n \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(P)$ with $A_i, \neg B_j \in M_1$ for all i and j. However, since $A \notin T^+_{P/G_P}\!\downarrow\!\gamma$ we obtain that for each $A \leftarrow A_1, \ldots, A_n$ in P/G_P, hence for each $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(P)$ with $\neg B_1, \ldots, \neg B_m \in \neg(B_P \setminus G_P) \subseteq M \subseteq M_1$ there is A_i with $A_i \notin T^+_{P/G_P}\!\downarrow\!(\gamma - 1) \subseteq M$, and by induction hypothesis we obtain $A_i \notin M_1$. So $A_i \in M_1$ and $A_i \notin M_1$, which is a contradiction and concludes the proof.

Definition 6. *For a normal program P, we call* $\mathsf{lfp}(\mathrm{CW}_P)$ *the* maximally circular well-founded model *(*maxwf model*) of P.*

5 Conclusions and Further Work

We have displayed a coherent picture of different semantics for normal logic programs. We have added to well-known results new ones which complete the formerly incomplete picture of relationships. The richness of theory and relationships turns out to be very appealing and satisfactory. From a mathematical perspective one expects major notions in a field to be strongly and cleanly interconnected, and it is fair to say that this is the case for declarative semantics for normal logic programs.

The situation becomes much more difficult when discussing extensions of the logic programming paradigm like disjunctive [22], quantitative [23], or dynamic [24] logic programming. For many of these extensions it is as yet to be determined what the best ways of providing declarative semantics for these frameworks are, and the lack of interconnections between the different proposals in the literature provides an argument for the case that no satisfactory answers have yet been found.

We believe that successful proposals for extensions will have to exhibit similar interrelationships as observed for normal programs. How, and if, this can be achieved, however, is as yet rather uncertain. Formal studies like the one in this paper may help in designing satisfactory semantics, but a discussion of this is outside the scope of our exhibition, and will be pursued elsewhere.

References

1. Hitzler, P., Wendt, M.: The well-founded semantics is a stratified Fitting semantics. In Jarke, M., Koehler, J., Lakemeyer, G., eds.: Proceedings of the 25th Annual German Conference on Artificial Intelligence, KI2002, Aachen, Germany, September 2002. Volume 2479 of Lecture Notes in Artificial Intelligence., Springer, Berlin (2002) 205–221
2. Hitzler, P., Wendt, M.: A uniform approach to logic rogramming semantics. Technical Report WV–02–14, Knowledge Representation and Reasoning Group, Artificial Intelligence Institute, Department of Computer Science, Dresden University of Technology, Dresden, Germany (2002) Submitted.
3. Reiter, R.: A logic for default reasoning. Artificial Intelligence **13** (1980) 81–132
4. Lloyd, J.W.: Foundations of Logic Programming. Springer, Berlin (1988)
5. Gelfond, M., Lifschitz, V.: The stable model semantics for logic programming. In Kowalski, R.A., Bowen, K.A., eds.: Logic Programming. Proceedings of the 5th International Conference and Symposium on Logic Programming, MIT Press (1988) 1070–1080
6. van Gelder, A., Ross, K.A., Schlipf, J.S.: The well-founded semantics for general logic programs. Journal of the ACM **38** (1991) 620–650
7. Fitting, M.: A Kripke-Kleene-semantics for general logic programs. The Journal of Logic Programming **2** (1985) 295–312
8. Apt, K.R., Blair, H.A., Walker, A.: Towards a theory of declarative knowledge. In Minker, J., ed.: Foundations of Deductive Databases and Logic Programming. Morgan Kaufmann, Los Altos, CA (1988) 89–148
9. Przymusinski, T.C.: On the declarative semantics of deductive databases and logic programs. In Minker, J., ed.: Foundations of Deductive Databases and Logic Programming. Morgan Kaufmann, Los Altos, CA (1988) 193–216
10. Fitting, M.: Fixpoint semantics for logic programming — A survey. Theoretical Computer Science **278** (2002) 25–51
11. Denecker, M., Marek, V.W., Truszczynski, M.: Approximating operators, stable operators, well-founded fixpoints and applications in non-monotonic reasoning. In Minker, J., ed.: Logic-based Artificial Intelligence. Kluwer Academic Publishers, Boston (2000) 127–144

12. Przymusinski, T.C.: Well-founded semantics coincides with three-valued stable semantics. Fundamenta Informaticae **13** (1989) 445–464
13. van Gelder, A.: The alternating fixpoint of logic programs with negation. In: Proceedings of the Eighth ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems, Philadelphia, Pennsylvania, ACM Press (1989) 1–10
14. Dung, P.M., Kanchanasut, K.: A fixpoint approach to declarative semantics of logic programs. In Lusk, E.L., Overbeek, R.A., eds.: Logic Programming, Proceedings of the North American Conference 1989, NACLP'89, Cleveland, Ohio, MIT Press (1989) 604–625
15. Fages, F.: Consistency of Clark's completion and existence of stable models. Journal of Methods of Logic in Computer Science **1** (1994) 51–60
16. Wendt, M.: Unfolding the well-founded semantics. Journal of Electrical Engineering, Slovak Academy of Sciences **53** (2002) 56–59 (Proceedings of the 4th Slovakian Student Conference in Applied Mathematics, Bratislava, April 2002).
17. Hitzler, P.: Circular belief in logic programming semantics. Technical Report WV–02–13, Knowledge Representation and Reasoning Group, Artificial Intelligence Institute, Department of Computer Science, Dresden University of Technology, Dresden, Germany (2002)
18. Jachymski, J.: Order-theoretic aspects of metric fixed-point theory. In Kirk, W.A., Sims, B., eds.: Handbook of Metric Fixed Point Theory. Kluwer Academic Publishers, Dordrecht, The Netherlands (2001) 613–641
19. Fitting, M.: Bilattices and the semantics of logic programming. The Journal of Logic Programming **11** (1991) 91–116
20. Abramsky, S., Jung, A.: Domain theory. In Abramsky, S., Gabbay, D., Maibaum, T.S., eds.: Handbook of Logic in Computer Science. Volume 3. Clarendon, Oxford (1994)
21. Przymusinska, H., Przymusinski, T.C.: Weakly stratified logic programs. Fundamenta Informaticae **13** (1990) 51–65
22. Wang, K.: A comparative study of well-founded semantics for disjunctive logic programs. In Eiter, T., Faber, W., Truszczynski, M., eds.: Logic Programming and Nonmonotonic Reasoning, 6th International Conference, LPNMR 2001, Vienna, Austria, September 17-19, 2001, Proceedings. Volume 2173 of Lecture Notes in Artificial Intelligence., Springer (2001) 133–146
23. Mateis, C.: Quantitative disjunctive logic programming: Semantics and computation. AI communications **13** (2000) 225–248
24. Leite, J.A.: Evolving Knowledge Bases. Volume 81 of Frontiers of Artificial Intelligence and Applications. IOS Press (2003)